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## Abstract

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# MATHEMATICS

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# NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBAL STABILITY FOR A SYSTEM OF THREE DIFFERENTIAL EQUATIONS

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In the present note we consider the system of three equations

$$\frac{dx}{dt} = y - ax - f(x), \qquad \frac{dy}{dt} = z - bf(x), \qquad \frac{dz}{dt} = -cf(x), \tag{1}$$

where the constants $a$, $b$, and $c$ satisfy the inequalities

$$ab > c, \qquad b > 0, \qquad c > 0, \tag{2}$$

and the continuous function $f(x)$ satisfies the condition for uniqueness of solutions of system (1) and the generalized Hurwitz condition

$$xf(x) > 0 \quad \text{for } x \neq 0; \qquad f(0) = 0. \tag{3}$$

System (1) under conditions (2) and (3) was considered by A. P. Tuzov ([1]), who indicated conditions sufficient for global stability of the zero solution. The aim of the present work is to establish necessary and sufficient conditions for global stability.

**Theorem.** *In order that the zero solution of system (1) be globally stable, it is necessary and sufficient that the following conditions be fulfilled:*

$$\lim_{x\to+\infty} \left( f(x) + \int_0^x f(x)\, dx \right) = +\infty, \tag{4}$$

$$\lim_{x\to-\infty} \left( -f(x) + \int_0^x f(x)\, dx \right) = +\infty. \tag{5}$$

The proof of sufficiency is based on the Lyapunov function constructed in ([1]) and on the theorem formulated in note ([2]).

Let us prove necessity. Suppose, for definiteness, that condition (4) is violated. Then there exists an $M > 0$ such that

$$f(x) < M \quad \text{for } x \geq 0. \tag{6}$$

Put

$$\int_0^{+\infty} f(x)\,dx = I < +\infty. \tag{7}$$

Let $H = a^2 - ab + c$.

First consider the case $H \geq 0$. Let the point $p$ of phase space have coordinates $x = 0,\ y = y_0,\ z = z_0$, and suppose that the inequalities

$$ay_0 - z_0 > 2aM + \frac{cl}{M}. \tag{8}$$

Through the point $p$ at $t = 0$ draw the trajectory $\varphi(p, t)$ of system (1). Next introduce for consideration the function

$$u = a^2x - ay + z. \tag{9}$$

The derivative of this function with respect to $t$, taken by virtue of system (1), has the form

$$\dot{u} = -au - Hf(x). \tag{10}$$

From equality (10) it follows that on $\varphi(p, t)$ the relation

$$u = e^{-at}\left(u_0 - H\int_0^t fe^{at}\,dt\right), \tag{11}$$

holds, where $u_0 = z_0 - ay_0$.

We shall show that on $\varphi(p, t)$, for $t \geq 0$, the inequality

$$z > 2aM \tag{12}$$

is satisfied.

For $t = 0$ this inequality is satisfied, as follows from (8). Suppose that it is violated at $t = t_1$. Then, by continuity, we may assume that $z(t_1) = 2aM$ and that for $t \in [0, t_1)$ inequality (12) is satisfied. We shall show that then on $\varphi(p,t)$, for $t \in [0, t_1]$, it will turn out that

$$y - ax > \frac{z}{a}. \tag{13}$$

Indeed, for $t = 0$ this inequality is satisfied. Suppose that there exists a $t^* \in (0, t_1]$ such that on the trajectory $\varphi(p,t)$ one has

$$y(t^*) - ax(t^*) = \frac{z(t^*)}{a},$$

and inequality (13) is satisfied for $t \in [0, t^*)$. But for $t \in [0, t_1]$, by assumption, $z \geq 2aM$; consequently,

$$y - ax \geq 2M > f(x) \tag{14}$$

on $\varphi(p,t)$ for $t \in [0, t^*]$. Hence, for such $t$, $x$ increases along $\varphi(p,t)$, and therefore for $t \in (0, t^*]$, $x > 0$ on $\varphi(p,t)$. But, by assumption, $H \geq 0$, whence from (11) it follows that for $t \in [0, t^*]$ on $\varphi(p,t)$, $u < 0$. And this means that for $t \in [0, t^*]$ (13) is satisfied. The contradiction obtained proves that inequality (13) is satisfied for all $t \in [0, t_1]$.

From inequalities (6), (12), and (13) it follows that for $t \in [0, t_1]$ on $\varphi(p,t)$ the relation

$$\frac{dz}{dx} > -\frac{cf(x)}{M}. \tag{15}$$

holds. Integrating this inequality, we obtain

$$z(t_1) - z_0 > -\frac{cl}{M},$$

and hence from (8) it follows that $z(t_1) > 2aM$, which contradicts the definition of the time $t_1$ ($z(t_1) = 2aM$). The contradiction obtained proves that inequality (12) is satisfied on the trajectory $\varphi(p,t)$ for all $t \geq 0$. Since $a > 0$ and $M > 0$, it follows from this that $\varphi(p,t)$ does not approach the origin. Thus, in the case under consideration, stability in the large is absent.

Let us now turn to the case $H < 0$. In this case let the point $p$ have coordinates $x = 0,\ y = y_0,\ z = z_0$, and let

$$z_0 = ay_0 > 2aM + (c - H)\frac{l}{M}. \tag{16}$$

We shall show that on the trajectory $\varphi(p,t)$ of system (1), for $t \geq 0$, the inequality

$$y - ax > 2M. \tag{17}$$

holds. For $t = 0$ this inequality is satisfied. Suppose that it is violated at $t = t_1 > 0$, so that

$$y(t_1) - ax(t_1) = 2M \tag{18}$$

and inequality (17) is satisfied for $t \in [0, t_1)$.

From inequalities (6) and (17) it follows that on the trajectory $\varphi(p,t)$, for $t \in [0, t_1]$, the relation

$$\frac{dz}{dx} > -\frac{cf(x)}{M}.$$

holds. Integrating this inequality, we obtain

$$z(t_1) - z_0 > -\frac{cI}{M},$$

whence, also from (16), we infer

$$z(t_1) > 2aM - \frac{HI}{M}. \tag{19}$$

Let us again turn to equality (10). From this equality, in view of (16), we obtain for the trajectory $\varphi(p,t)$

$$u = -He^{-at} \int_0^t fe^{at}\, dt; \tag{20}$$

hence, and from (17), just as in the case $H \geq 0$, we establish that for $t \in [0, t_1]$ on the trajectory $\varphi(p,t)$ one has $u \geq 0$. Dividing equality (10) by the first of the equations of system (1), we then obtain

$$\frac{du}{dx} = \frac{-au - Hf(x)}{y - ax - f(x)};$$

hence, from $u \geq 0$, we obtain the inequality

$$\frac{du}{dx} \leq -\frac{Hf(x)}{y - ax - f(x)},$$

which is valid on $\varphi(p,t)$ for $t \in [0,t_1]$. The last inequality, in view of (6) and (17), gives

$$\frac{du}{dx} \le -\frac{Hf(x)}{M}.$$

Integrating this inequality along the trajectory $\varphi(p,t)$ from $t=0$ to $t=t_1$, we obtain

$$u(t_1) - u(0) < -\frac{HI}{M}.$$

But from (16) it follows that $u(0)=0$. Therefore, from the last inequality and from (9), we obtain

$$a\,[ax(t_1) - y(t_1)] + z(t_1) < -\frac{HI}{M}.$$

Hence, and from (9), we infer

$$y(t_1) - ax(t_1) > 2M,$$

which contradicts the definition of the time $t_1$ (equality (18)). The contradiction obtained proves that on the trajectory $\varphi(p,t)$, for $t \ge 0$, inequality (17) is satisfied. It follows that the trajectory $\varphi(p,t)$ does not tend to the origin as $t \to +\infty$. Consequently, in this case as well there is no stability in the large. The necessity is proved.

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## REFERENCES

[1] A. P. Tuzov, *Vestn. LGU*, No. 2 (1955). [2] V. A. Pliss, *DAN*, **101**, No. 1 (1955).

*Note: Figure translations are in progress. See original paper for figures.*