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Abstract

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CUBIC RESOLVENTS

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In this note a theory is developed which is a "cubic" analogue of the theory ([1]) of natural systems (or, as we now prefer to say, of homotopy resolvents). In many questions connected with the study of various fiberings, the "cubic resolvents" constructed in the present note are considerably more convenient than homotopy resolvents (which have a "simplicial" character). Applications of cubic resolvents to the theory of fiber spaces will be set forth in subsequent communications.

1. Cubic complexes

(see, for example, ([2])). We consider a set K of certain elements σ, called cubes, to each of which is assigned a certain integer $n \geq 0$, called its dimension. It is assumed that to every n-dimensional cube σ there correspond $2n$ cubes $\sigma 0^i$ and $\sigma 1^i$, $1 \leq i \leq n$, of dimension $n-1$, and $n+1$ cubes $\sigma \eta^j$, $1 \leq j \leq n+1$, of dimension $n+1$. The cube $\sigma 0^i$ is called the i-th lower face of the cube σ, the cube $\sigma 1^i$ its i-th upper face, and the cube $\sigma \eta^j$ is called the j-th degeneracy of the cube σ. If the operators 0^i, 1^i, and η^j satisfy the relations

$$\varepsilon^i \omega^j = \begin{cases} \omega^{j+1}\varepsilon^i, & \text{if } i \leq j, \\ \omega^j \varepsilon^{i-1}, & \text{if } i > j; \end{cases} \qquad \eta^i \eta^j = \begin{cases} \eta^{j-1}\eta^i, & \text{if } i < j, \\ \eta^j \eta^{i+1}, & \text{if } i \geq j; \end{cases}$$

$$\varepsilon^i \eta^j = \begin{cases} \eta^{j+1}\varepsilon^i, & \text{if } i < j, \\ \eta^{i+1}\varepsilon^i = \eta^i \varepsilon^{i+1}, & \text{if } i = j, \\ \eta^j \varepsilon^{i+1}, & \text{if } i > j; \end{cases} \qquad \eta^i \varepsilon^j = \begin{cases} \varepsilon^{j-1}\eta^i, & \text{if } i < j, \\ 1, & \text{if } i = j, \\ \varepsilon^j \eta^{i-1}, & \text{if } i > j, \end{cases}$$

where $\varepsilon = 0, 1$, $\omega = 0, 1$, and 1 is the identity mapping, then the set K is called a cubic complex. A mapping $L \to K$ of cubic complexes is called a cubic mapping if it preserves dimensions and commutes with the operators 0^i, 1^i, and η^j. A complex L contained in a complex K is called a subcomplex of the complex K if the inclusion mapping $L \to K$ is a cubic mapping. The smallest subcomplex of the complex K containing all its nondegenerate (i.e., not representable in the form $\sigma\eta^j$) cubes of dimensions $\leq n$ is called the n-dimensional skeleton of the complex K and is denoted by the symbol K^n. The direct product of two cubic complexes K and L is the complex $K \times L$, whose cubes of dimension n are the pairs (σ, τ), where σ, τ are arbitrary n-dimensional cubes of the complexes K

and L, respectively, and in which the operators ε^i, $\varepsilon = 0, 1$, and η^j are defined by the formulas $(\sigma, \tau)\varepsilon^i = (\sigma\varepsilon^i, \tau\varepsilon^i)$, $(\sigma, \tau)\eta^j = (\sigma\eta^j, \tau\eta^j)$.

2. Cochains in cubic complexes

Let K be an arbitrary cubic complex and G some group (multiplicative or additive). A G-valued function c, defined on the set of all n-dimensional cubes of the complex K, is called an n-dimensional cochain of the complex K over the group G if, on every degenerate (i.e., having the form $\sigma\eta^j$) cube of the complex K, its value is equal to the neutral element of the group G. The totality of all n-dimensional cochains of the complex K over the group G (which is naturally a group) is denoted by the symbol $C^n(K; G)$. Let us note that any cubic mapping $f : L \to K$ induces (by the known formula $(f^*c)(\sigma) = c(f\sigma)$) a certain homomorphism

$f^* : C^n(K; G) \to C^n(L; G)$. Similarly, any homomorphism $\theta : G \to H$ induces (by the formula $(\theta^0 c)(\sigma) = \theta(c(\sigma))$) a homomorphism $\theta^0 : C^n(K; G) \to C^n(K; H)$.

3. Cocyles over multiplicative groups. Let the group G be a multiplicative group (i.e., an arbitrary group written multiplicatively). A one-dimensional cochain a^1 of a cubical complex K over the group G will be called a **cocycle** if

$$a^1(\sigma 0^1) a^1(\sigma 1^2) = a^1(\sigma 0^2) a^1(\sigma 1^1)$$

for every two-dimensional cube $\sigma \in K$. It is obvious that the homomorphisms f^* and θ^0 take cocycles to cocycles. (However, the set of all cocycles of the complex K over the group G, generally speaking, is not a subgroup of the group $C^1(K; G)$.)

4. Cohomology over additive groups. Let now G be an additive group (i.e., an Abelian group written additively), on which some multiplicative group G_1 acts (as a group of left operators). Then every cocycle a^1 of the complex K over the group G_1 allows one to define on the groups $C^n(K; G)$ a certain coboundary operator ∇_{a^1}:

$$(\nabla_{a^1} c^n)(\sigma) = \sum_{i=1}^{n+1} (-1)^i (c^n(\sigma 0^i) - a^1(\sigma 0^{n+1} \dots \hat{0}^i \dots 0^1) c^n(\sigma 1^i)),$$

where $c^n \in C^n(K; G)$, and σ is an arbitrary $(n+1)$-dimensional cube of the complex K. The homology groups of the cochain complex $\{C^n(K; G); \nabla_{a^1}\}$ are called the **cohomology groups** $H^n_{a^1}(K; G)$ **of the cubical complex** K **over the group** G **relative to the cocycle** a^1. In particular, cochains c^n for which $\nabla_{a^1} c^n = 0$ are called **cocycles relative to** a^1. Cohomology classes will be denoted by boldface letters. For any cubical map $f : L \to K$, the homomorphisms f^* commute with the operators ∇_{a^1} (in the sense that $f^* \nabla_{a^1} = \nabla_{f* a^1} f^*$). Therefore an induced homomorphism arises:

$$f^* : H^n_{a^1}(K; G) \to H^n_{f* a^1}(L; G).$$

Similarly, $\theta^0\nabla_{a^1} = \nabla_{\theta^0 a^1}\theta^0$, for any θ-homomorphism $\vartheta : G \to H$, where $\theta : G_1 \to H_1$ (here H is an additive group on which the multiplicative group H_1 acts). Consequently, the θ-homomorphism ϑ induces a homomorphism

$$\vartheta^0 : H^n_{a^1}(K;\, G) \to H^n_{\theta^0 a^1}(K;\, H).$$

5. Complexes $E(G,p)$. The set of all faces of the unit cube of n-dimensional arithmetic space (together with the cube itself) is naturally defined as the set of all nondegenerate cubes of a certain cubical complex I^n (the operators of the expressions η^i in the complex I^n are defined formally, and the boundary operators ε^i on the basis of the geometric incidence relations). Cochains of dimension p of the complex I^n over a certain group G will be called **cubical tuple-functions over the group G of height p and dimension n**. Projection "along the j-th axis," where $1 \leq j \leq n+1$, naturally defines a certain cubical map $I^{n+1} \to I^n$. The map induced by this map

$$C^p(I^n;\, G) \to C^p(I^{n+1};\, G)$$

will be denoted by η^j. Similarly, by setting equal to zero or one the i-th, $1 \leq i \leq n$, coordinate (of the arithmetic space under consideration), we obtain in the complex I^n two subcomplexes isomorphic to the complex I^{n-1}. The maps induced by the corresponding embedding maps

$$C^p(I^n;\, G) \to C^p(I^{n-1};\, G)$$

will be denoted by 0^i and 1^i, respectively. It is readily verified that the set $E(G,p)$ of all cubical tuple-functions of height p over the group G is, relative to the operators 0^i, 1^i, and η^i, a cubical complex, and its subset $Q(G,p)$, consisting of tuple-functions that are cocycles, is its subcomplex. The cubes of dimension p of the complex $Q(G,p)$ are in a natural one-to-one-

one-to-one correspondence with the elements of the group G. Assigning to each p-dimensional cube the corresponding element of the group G, we obviously obtain a certain cochain 1^p_G of the complex $Q(G,p)$ over the group G.

6. **Extensions of cubic complexes.** Let K be an arbitrary cubic complex; G an additive group on which the multiplicative group G_1 acts; a^1 a certain cochain of the complex K over the group G_1, and k^{p+1} an arbitrary $(p+1)$-dimensional $(p > 1)$ cochain of the complex K over the group G relative to a^1. Consider the subset K' of the direct product $K \times E(G,p)$, consisting of all pairs (σ, c), $\sigma \in K$, $c \in E(G;p)$, for which

$$\nabla_{f_\sigma{}^* a^1} c + f_\sigma^* k^{p+1} = 0,$$

where f_σ is the cubic mapping $I^n \to K$ that sends the unique nondegenerate n-dimensional cube of the complex I^n to the cube σ (by this condition the mapping f_σ is determined uniquely). As may be verified, the subset K' is a subcomplex of the complex $K \times E(G,p)$. We shall call it the

p-extension of the complex K over the group G with factor k^{p+1} relative to the cochain a^1. It is easy to see that, for any cube $\sigma \in K$ of dimension $\leq p$, the pair $(\sigma, 0)$, where 0 is the zero tuple-function, belongs to the p-extension K'. Therefore, identifying σ with $(\sigma, 0)$, we obtain that $K^p \subset K'^p$. At the same time, as is easy to see, $K^{p-1} = K'^{p-1}$. Hence, in particular, it follows that the cochain a^1 may also be regarded as a cochain of the complex K'. By definition, the extension K' depends on the factor k^{p+1}. However, it turns out that extensions with cohomologous factors are isomorphic. Thus the complex K' depends, in essence, only on the cohomology class of the factor k^{p+1}.

7. **Cubic resolvents.** A sequence $K_1, K_2, \ldots$ of cubic complexes will be called a resolvent sequence if $K_1 = Q(G_1, 1)$, where G_1 is a certain multiplicative group, and for every $i > 1$ the complex K_i is an i-extension of the complex K_{i-1} with some factor k_{i-1} over a certain additive group G_i (on which the group G_1 acts) relative to the cochain $1^1_{G_1}$ (regarded as a cochain of the complex K_{i-1}). Since in a resolvent sequence $K^i_{i-1} \subset K^i_i$, a limiting complex K is defined, for which $K^i = K^i_i$. The system $\{G_i, k_i\}$ of groups G_i and factors k_i will be called a cubic resolvent of the complex K. Let $\{G_i, k_i\}$, $\{H_i, l_i\}$ be two resolvents and $\{K_i\}$, $\{L_i\}$ the corresponding resolvent sequences. Suppose that for every $i \geq 1$ a homomorphic mapping $\theta_i : G_i \to H_i$ is given, which for $i > 1$ is a θ_1-homomorphism. The mapping θ_1 naturally induces a certain cubic mapping $\bar{\theta}_1 : K_1 \to L_1$. Suppose that, for some $i \geq 1$, a cubic mapping $\bar{\theta}_i : K_i \to L_i$ has already been constructed. It turns out that if $\bar{\theta}^*_i l_i = \theta^0_{i+1} k_i$, then there exists a cubic mapping $\bar{\theta}_{i+1} : K_{i+1} \to L_{i+1}$ that coincides on K^{i+1}_i with the mapping $\bar{\theta}_i$. If the relations $\bar{\theta}^*_i l_i = \theta^0_{i+1} k_i$ are fulfilled successively for all $i \geq 1$, then the system of homomorphisms $\{\theta_i\}$ will be called a homomorphism of the resolvent $\{G_i, k_i\}$ into the resolvent $\{H_i, l_i\}$. If all the mappings θ_i are isomorphisms, then the homomorphism $\{\theta_i\}$ will be called an isomorphism. It turns out that the limiting complexes of two resolvent sequences are isomorphic if and only if the corresponding resolvents are isomorphic.

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REFERENCES

1. M. M. Postnikov, *Tr. Matem. inst. im. V. A. Steklova AN SSSR*, **46**, 1 (1955).

2. D. M. Kan, *Proc. Nat. Acad. Sci. USA*, **41**, 1092 (1955).

Note: Figure translations are in progress. See original paper for figures.