Mathematics

Ch. Foiaș, G. Gussi, and B. Poénaru

1958

SovietRxiv

Abstract

Full Text

Mathematics

Ch. Foiaș, G. Gussi, and B. Poénaru

GENERALIZED SOLUTIONS OF A QUASI-LINEAR DIFFERENTIAL EQUATION IN A BANACH SPACE

(Presented by Academician S. L. Sobolev on 26 XI 1957)

1. Consider the equation

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1}$$

in a Banach space X, where $A(t)$ are linear closed operators with domains $D_{A(t)}$ dense in X, satisfying the condition:

C I. There exist sequences J_n of idempotent operators such that $J_{nJ}m = 0$ for $n \neq m$ and, moreover:

1) $I = J_1 + J_2 + \dots + J_n + \dots$ (the series converges strongly);
2) $J_{nA}(t) \subseteq A(t)J_n$ $(t \in (a, b))$, $n = 1, 2, \dots$;
3) $A(t)J_n$ is bounded in $X_{(n)} = J_n(X)$.
4) $A(t)J_n$ is continuous in norm in $X_{(n)}$ for $t \in (a, b)$, $n = 1, 2, \dots$.

It is possible that the Cauchy problem for equation (1) has no solution for all initial conditions $y_0 \in X$. Consider the space $\mathfrak{X}$ of sequences $x = \{x_n\}$, where $x_n \in X_{(n)}$, with the family of seminorms

$$p_{n_1 n_2 \dots n_k}(x) = \sup(\|x_{n_1}\|, \dots, \|x_{n_k}\|),$$

where $(n_1, \dots, n_k)$ is any finite set of natural numbers; the correspondence $x \to \{J_{nx}\} = \tilde{x}$ is a continuous isomorphism from X onto $\widetilde{X} \subset \mathfrak{X}$ ($\mathfrak{X}$ is the closure of $\widetilde{X}$).

Associate with equation (1) the equation

$$\frac{dx}{dt} = \widetilde{A}(t)x(t), \tag{1'}$$

where $\widetilde{A}(t)\{x_n\} = \{A(t)x_n\}$ is the extension of $A(t)$ in $\mathfrak{X}$ by continuity.

Theorem 1. 1) The Cauchy problem for equation $(1')$ has a solution, and moreover a unique one in (a, b), for arbitrary initial data from $\mathfrak{X}$.

2) The Cauchy problem is well posed in $\mathfrak{X}$, i.e., on any closed interval contained in (a, b) the solution depends continuously on the initial value uniformly with respect to t.

3) If $x(t)$ is a solution of the Cauchy problem for (1), $x(c) = x_0$, then $\tilde{x}(t)$ is a solution of the Cauchy problem for $(1')$, $\tilde{x}(c) = \tilde{x}_0$.

It follows from this that the Cauchy problem for (1) has at most one solution and that this solution depends continuously on the initial data, if it is considered in $\mathfrak{X}$. If the initial value belongs to $\widetilde{X}$, then the solution of the Cauchy problem for $(1')$ is a generalized solution for (1).

We introduce the following condition:

C II. There exists a function $\gamma(t)$, summable on any interval $[a_1, b_1] \subset (a, b)$, such that for any $\varepsilon > \gamma(t)$ the operator $[\varepsilon I - A(t)]^{-1}$ exists,

defined on all of X and satisfies the inequality*

$$\left\|[\varepsilon I - A(t)]^{-1}\right\| < \frac{1}{\varepsilon - \gamma(t)}, \qquad \varepsilon > \gamma(t), \quad t \in (a, b).$$

Theorem 2. If $A(t)$ satisfies condition C II, then:

1) The solution of the Cauchy problem for $(1')$ takes its values in $\widetilde{X}$ for initial values from $\widetilde{X}$.

2) The solution is continuous in X for $t \geq c$.

3) If $\tilde{x}(t)$ is a solution of the Cauchy problem for $(1')$ such that $\tilde{x}(c) = \tilde{x}_0$, $x_0 \in X$, then for $t \geq c$

$$\|x(t)\| \leq \|x_0\| \exp\left[\int_c^t \gamma(\tau)\, d\tau\right].$$

Corollary 1. 1) The solution of the Cauchy problem for $(1')$ with initial conditions in X and considered in X is a generalized solution of the Cauchy problem for (1) in the sense of S. L. Sobolev ([2]).

2) The generalized Cauchy problem is well posed.

Denote by $T(t, s)x$ $(t \geq s)$ the generalized solution of the Cauchy problem such that $T(t, s)x \to x$ as $t \to s + 0$.

Corollary 2. 1) $T(t, s)T(s, \gamma) = T(t, \gamma)$, $t \geq s \geq \gamma$.

2) $T(t, s)$ is strongly continuous on the triangle $t, s \in (a, b)$, $t \geq s$.

3)

$$\|T(t,s)\| \leq \exp\left[\int_s^t \gamma(\tau)\,d\tau\right].$$

4) If

$$x_0 \in \bigcup_{N=1}^{\infty} \sum_{n=1}^{N} X_{(n)},$$

then $T(t,s)x_0$ is a solution of the Cauchy problem for (1).

In Kato' s construction ([1]), the set $\{x\}$ for which $T(t,s)x$ is a solution of (1) coincides with $D_{A(0)} = D_{A(t)}$. In our case it is possible that $D_{A(t)} \neq D_{A(0)}$, but always

$$\bigcup_{N=1}^{\infty} \sum_{n=1}^{N} X_{(n)} \subset D_{A(0)},$$

and the inclusion is often strict.

We introduce the following condition:

C III. Let condition C II be satisfied and let $\gamma(t) \leq \gamma < \infty$, and let γ_n be the least number such that:

1) $[\varepsilon I - A_{(n)}(t)]^{-1}$ exists in $X_{(n)}$ and is defined on all $X_{(n)}$ for $\varepsilon > \gamma_n$ ($A_n(t)$ is the restriction of $A(t)$ to $X_{(n)}$).

2)

$$\left\|[\varepsilon I - A_{(n)}(t)]^{-1}\right\| \leq \frac{1}{\varepsilon - \gamma_n}, \qquad t \in (a,b).$$

Let, moreover,

$$M_{(n)} = \max_{t \in (a,b)} \|A_{(n)}(t)\|.$$

Then

$$\sum_{n=1}^{\infty} M_{(n)} e^{t\gamma_n} < \infty.$$

There exist operators satisfying these conditions. For example, if X is a Hilbert space and $A(t) = B(t)N$, where N is a normal unbounded operator whose spectrum lies in an angle smaller than π, with the negative half-axis Ox serving as the bisector of this angle, and $B(t)$ is a strictly positive operator, continuous in norm and such that $B(t)N \subset NB(t)$, then condition C III is satisfied ([3]).

Theorem $1'$**.** If condition C III is satisfied, $T(t,s)x$ is a solution of the Cauchy problem for (1) for $x \in X$, $t > s$.

This result is analogous to a property that holds for certain parabolic equations of I. G. Petrovskii ([4]).

2. Consider the equation

$$\frac{dx}{dt} = A(t)x + f(t,x). \tag{2}$$

* The case $\gamma(t) = \gamma$, see (1).

Suppose that $A(t)$ satisfies condition C II and that $f(t,x)$ is a continuous mapping of $(a,b) \times X$ into X. Then every solution of the Cauchy problem for (2), $x(s) = x_0,\ x_0 \in X$, will be a solution* of the equation

$$x(t) = T(t,s)x_0 + \int_s^t T(t,\tau)f(\tau, x(\tau))\,d\tau, \tag{3}$$

and, conversely, every solution of this equation is a solution of the Cauchy problem for the equation

$$\frac{d\tilde{x}(t)}{dt} = \widetilde{A}(t)\tilde{x}(t) + \tilde{f}(t,\tilde{x})$$

from $\mathfrak{X}$ (obviously, $\tilde{f}$ is defined only on $(a,b) \times \widetilde{X}$). It is natural to regard a solution of (3) as a generalized solution for (2).

Theorem 3. Let $s \in (a,b)$, and let S be a ball in X with center at x_0. If $f(t,x)$ is defined and continuous on $(a,b) \times S$ and $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1(t,x)$ is compact and

$$\|f_2(t,x) - f_2(t,y)\| \le \overline{K}(t)\omega(\|x-y\|);$$

$\overline{K}(t)$ is summable on (a,b) and $\omega(z)$ is an Osgood-type function ([6,7]), then there exists an interval $[s,b) \subset (a,b)$ on which (3) has at least one solution $x(s) = x_0$. If $f_1 \equiv 0$, this solution is unique.

In the proofs the following lemma was used.

Lemma 1. Let C be the space of continuous functions on $[0,T]$ with values in X, and let F be a mapping of a bounded closed set $M \subset C$ into itself, satisfying, for $u, v \in M$, the condition

$$\|Fu(t) - Fv(t)\| \leq \int_0^t K(\tau)\omega(\|u(\tau) - v(\tau)\|)\, d\tau,$$

where $K(t)$ is summable on $[0, T]$, and $\omega(z)$ is an Osgood-type function. Then the equation $u = Fu$ has a solution in M, and moreover a unique one**.

Theorem 4. Under the hypotheses of Theorem 3, if $f(t, u)$ satisfies on $(a, b) \times X$ the condition

$$\|f(t, x)\| \leq \overline{K}(t)\overline{\omega}(\|x\|),$$

where $\overline{K}(t)$ is summable on every interval $[a_1, b_1] \subset (a, b)$ and $\overline{\omega}(z)$ is a Wintner-type function, then the solution of equation (3) can be continued to all of $[c, b)$ (b may be infinite).

We note that Theorems 3 and 4 also hold when $T(t, s)$ satisfies conditions 1) and 2) of Corollary 2, and therefore in this case they also include the generalized solutions from ([5]).

For simplicity suppose that in condition C II $\gamma(t) = \gamma$. Then:

Corollary 3. If, under the hypotheses of Theorem 4, $\overline{K}(t)$ is summable on (a, b), then:

1) If $b < \infty$, the solution is uniformly bounded on $[c, b]$.

2) If $b = \infty$ and $\gamma \leq 0$, the solution is uniformly bounded on $[c, \infty)$.

3) If $b = \infty$ and $\gamma < 0$, the solution tends to zero as $t \to \infty$ (whatever the initial conditions may be).

Corollary 4. 1) If, under the hypotheses of Theorem 4, $f_1 \equiv 0$, then the Cauchy problem is well posed.

2) If, in addition, $K(t)$ is summable on all of (a, b), then the generalized Cauchy problem is well posed uniformly with respect to $c \in [a_1, b_1] \subset (a, b)$.

3) If $b = \infty$ and $\gamma \leq 0$, the solution of the generalized Cauchy problem is Lyapunov stable.

---

* This does not always hold under Kato' s conditions ([5]).

** This lemma makes it possible to extend Krasnosel' skii' s fixed-point theorem ([8]) and is proved with the aid of Lemma I. 1 from ([6]); the latter, in a more particular form, was also obtained by Krasnosel' skii ([9]).

3. Suppose that $A(t) = A$.

Theorem 5. *Let the hypotheses of Theorem 3 and the following conditions be satisfied:*

1) *For* $u \in D_A$*,* $f(t,u) \in D_A$*.*

2) $Af(t,u(t))$ *is continuous for every continuous* $u(t) \in D_A$*.*

3) $\|Af(t,u) - Af(t,v)\| \leq K_1(t)\omega_1(\|Au - Av\|)$*, where* $u, v \in D_A$*,* $K_1(t)$ *is summable, and* $\omega_1(z)$ *is a function of Osgood type.*

Then every solution of the generalized Cauchy problem for (2) is a solution in the usual sense.

With slight changes the theorem remains valid also if $A(t)$ depends on t, but the Kato conditions ([1]) are satisfied; however, as is easy to see ([5]), such a theorem is ineffective.

Theorem 6. *Let, under the hypotheses of Theorem 2:*

1) $f_t(t,u)$ *exist and be continuous.*

2) $f(t,u)$ *have a continuous Fréchet differential* $f_u(t,u)$*.*

Then the Cauchy problem for (2) has a unique solution on a sufficiently small interval for initial conditions from X*.*

This theorem is an improvement of the theorem from ([5]) in the case where $A(t)$ does not depend on t. Theorems 5 and 6 also hold in the case where $T(t)$ is a strongly continuous subgroup and $T(t) \to I$, consequently, if A satisfies the general condition of Miyadera–Phillips ([10]).

Received
7 X 1957

REFERENCES

[1] T. Kato, J. Math. Soc. Japan, **5**, No. 2 (1953).
[2] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[3] S. Foiaş, G. Gussi, V. Poenaru, Trans. Am. Math. Soc. **89**, No. 2 (1957).
[4] I. G. Petrovskii, Bull. Moscow State Univ., Sect. A, **1**, No. 7 (1938).
[5] M. A. Krasnosel' skii, S. P. Krein, P. E. Sobolevskii, DAN, **111**, No. 1 (1956).
[6] S. Foiaş, G. Gussi, V. Poenaru, Math. Nachr., **15**, H. 2 (1956).
[7] G. Gussi, V. Poenaru, K. Foiaş, DAN, **112**, No. 3 (1957).
[8] M. A. Krasnosel' skii, Uspekhi Mat. Nauk, **10**, issue 1 (63) (1955).
[9] M. A. Krasnosel' skii, Proc. Seminar on Functional Analysis, Voronezh State Univ., 1957.
[10] I. Miyadera, Tôhoku Math. J., (2), **4** (1952).

Note: Figure translations are in progress. See original paper for figures.