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Abstract

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HYDROMECHANICS

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ON THE PROPAGATION OF DISTURBANCES IN MEDIA WITH A NONLINEAR DEPENDENCE OF STRESSES ON DEFORMATIONS AND TEMPERATURE

(Presented by Academician L. I. Sedov on 11 I 1958)

The problem of the propagation of disturbances by plane waves in media with a nonlinear dependence of stresses only on deformations, $\sigma = \sigma(e)$, in the case when $\partial^2\sigma/\partial e^2 > 0$, was considered by Kh. A. Rakhmatulin ([1]). Self-similar motions, including the case when $\partial^2\sigma/\partial e^2$ changes sign, likewise without taking temperature changes into account, were considered by G. I. Barenblatt ([2,3]).

In adiabatic dynamic processes, the change in the temperature of particles may occur both as a consequence of continuous deformation and as a result of the irreversible transition of mechanical energy into heat at surfaces of strong discontinuity, and therefore allowance for thermal effects is necessary. The extent to which thermal effects influence mechanical phenomena depends on the nature of the substance; in each individual case this question must be investigated specially.

Below, under the assumption of an arbitrary dependence of stresses on deformations and temperature, a solution is given of the dynamic problem of the propagation of disturbances in a medium occupying the half-space $x \geqslant 0$, arising under the action of a constant stress applied to the boundary of the medium. The investigation is based on the method of dimensions developed by L. I. Sedov ([3]). This method and previously obtained results ([7]) make it possible to solve, for ideal compressible media (gas, liquid, solids), with an arbitrary equation of state and with allowance for thermal effects, a large class of problems in which the elementary work of the internal forces is represented in the form of a single term $Y\,dy$. In particular, one-dimensional motions, longitudinal extension and compression of rods, and problems of longitudinal and transverse impact on a flexible deformable string may be considered. Some of these problems, as is readily seen, fit into the formulation given in the present communication. A number of questions and problems for a gas with an arbitrary equation of state, but under the condition $(\partial^2\sigma/\partial e^2)_S < 0$, are considered in the book by L. D. Landau and E. M. Lifshitz ([4]). Below this restriction on the sign of the second derivative is removed.

§ 1. Suppose that the internal energy of the medium E, as a function of the deformation e and entropy S, is known. By partial differentiation of the function $E = E(e, S)$, according to the equation

$$dE = T\,dS + \frac{1}{\rho_0}\sigma\,de$$

we obtain the stress σ and the absolute temperature T as functions of deformation and entropy: $\sigma = \sigma(e, S)$, $T = T(e, S)$. The integrability condition gives

$$\frac{1}{\rho_0}(\partial\sigma/\partial S)_e = (\partial T/\partial e)_S.$$

It is known from thermodynamics ([5]) that, for stable thermodynamically equilibrium states, one must have

$$(\partial S/\partial T)_e > 0, \quad (\partial\sigma/\partial e)_T > 0, \quad (\partial\sigma/\partial e)_S > 0.$$

The system of equations of adiabatic motions with plane waves has the form

$$\frac{\partial\sigma}{\partial x} = \rho\frac{\partial^2 x}{\partial t^2}, \qquad \frac{\rho_0}{\rho} = \frac{\partial x}{\partial x_0} = 1 + e, \qquad \frac{\partial S}{\partial t} = 0; \tag{1}$$

here ρ is the density of the medium, t is time; the subscript 0 denotes the initial values of the parameters.

We shall construct a solution of system (1) satisfying the following initial and boundary conditions

$$\begin{aligned} &t = 0: \quad x = x_0, \quad \frac{\partial x}{\partial t} = 0, \quad S = S_0, \quad \sigma = \sigma(0, S_0) = \sigma_0, \quad T = T(0, S_0) = T_0; \\ &x_0 = 0: \quad \sigma = A = \text{const} \neq \sigma_0, \quad A < 0. \end{aligned} \tag{2}$$

Without restricting generality, put $(\partial\sigma/\partial S)_e < 0$, $\sigma_0 = 0$.

It follows from dimensional theory that the motion under study is self-similar and that all dimensionless quantities may be sought in the form of functions of one independent dimensionless variable ([3]):

$$\lambda = \frac{x_0}{t\sqrt{\sigma_k/\rho_0}}, \qquad \frac{x}{t\sqrt{\sigma_k/\rho_0}} = X(\lambda),$$

$$\frac{S}{S_k} = \eta(\lambda), \qquad e = e(\lambda), \qquad \frac{\sigma}{\sigma_k} = P(e,\eta), \qquad \frac{T}{T_k} = \tau(e,\eta), \qquad \frac{E}{E_k} = U(e,\eta),$$

where $\sigma_k > 0$; T_k are certain values of stress and temperature characteristic for the given medium;

$$E_k = \frac{\sigma_k}{\rho_0}, \qquad S_k = \frac{\sigma_k}{\rho_0 T_k}.$$

Passing in (1), (2) to dimensionless variables, we obtain:

$$\left[\lambda^2 - \left(\frac{\partial P}{\partial e}\right)_\eta\right] X''(\lambda) = 0, \qquad X'(\lambda) = 1 + e, \qquad \eta'(\lambda) = 0; \qquad (1')$$

$$\lambda = \infty: \qquad e = 0, \qquad P = 0, \qquad \frac{X}{\lambda} = 1, \qquad \eta = \frac{S_0}{S_k} = \eta_1, \qquad \tau = \frac{T_0}{T_k} = \tau_1;$$

$$\lambda = 0: \qquad P = \frac{A}{\sigma_k} = \beta < 0. \qquad (2')$$

On the surface of a strong discontinuity the conditions ([3,6]) must be satisfied:

$$\lambda_1^* = \lambda_2^* = \lambda^* = \text{const}, \qquad \lambda^{*2} = \frac{P_2 - P_1}{e_2 - e_1}, \qquad U_2 - U_1 = \frac{P_1 + P_2}{2}(e_2 - e_1). \quad (3)$$

The condition $S_2 \geq S_1$ is fulfilled if the inequalities ([7])

$$\left\{\frac{P_1 + P_2}{2}(e_2 - e_1) - \int_{e_1}^{e_2} P\,\Big|_{\eta=\eta_1} de\right\} \geq 0, \qquad \left\{\frac{P_1 + P_2}{2}(e_2 - e_1) - \int_{e_1}^{e_2} P\,\Big|_{\eta=\eta_2} de\right\} \geq 0. \quad (4)$$

The form and properties of the Hugoniot adiabats for each of the Poisson adiabat fields considered below are established by means of known methods, as well as the results set forth in ([7]).

§ 2. The system of equations ($1'$) has two solutions. One corresponds to motion with constant velocity for constant values of strain, entropy, stress, and temperature. The second solution represents a nonstationary compression (tension) wave; for the surfaces limiting it, $\lambda^2 = (\partial P/\partial e)_\eta = \text{const}$. Obviously, adjacent

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

to the boundary of the medium ($x_0 = 0,\ \lambda = 0$), by virtue of the condition $(\partial P/\partial e)_\eta = \theta(e, \eta) > 0$, there will be a region of translational motion.

1°. If along the Poisson adiabats $(\partial^2 P/\partial e^2)_\eta < 0$ (Fig. 1), then, as follows from inequalities (4), only compression shock waves are possible, with

$$\theta(e_2, \eta_2) > \lambda^{*2} = \frac{P_2 - P_1}{e_2 - e_1} > \theta(e_1, \eta_1),$$

and in the direction $x_0 > 0$ there cannot simultaneously follow a compression shock wave and a nonstationary tension wave, nor two compression shock waves separated by a region of stationary–

···of the stressed state of the medium. The solution has the form

$$e = 0, \qquad \eta = \eta_1, \qquad P = 0, \qquad \tau = \tau_1, \qquad \frac{X}{\lambda} = 1 \qquad \left(\lambda \geqslant \lambda^* = \sqrt{\frac{\beta}{e_\beta}}\right);$$

$$P = \beta, \qquad e = e_\beta, \qquad \eta = \eta_\beta, \qquad \tau = \tau(e_\beta, \eta_\beta), \qquad X = (1+e_\beta)\lambda - e_\beta\lambda^* \qquad (0 \leqslant \lambda \leqslant \lambda^*);$$

e_β is the abscissa of the point of intersection of the shock adiabat Γ_1 with the straight line $P = \beta$.

2°. Suppose that in the region $e < 0$ the derivative $(\partial^2 P/\partial e^2)_\eta$ vanishes once and changes sign while increasing (Fig. 2). The curve BK is the locus of points of intersection of shock adiabats having, as "centers," points of the arc OB of the adiabat $\eta = \eta_1$, with the corresponding tangents constructed at the "centers." This curve is intersected by the horizontals $P = \beta$ ($P_3 \leqslant \beta \leqslant P_1$) at one point.

Fig. 1

Fig. 2

A. $P_1 \leqslant \beta < 0$. Analysis shows that ahead of the region of progressive motion, forming at the boundary of the medium, there will travel a nonstationary compression wave whose boundaries are weak-discontinuity surfaces.

For $\lambda \geqslant \lambda_1 = \sqrt{\theta(0, \eta_1)}$:

$$e = 0, \qquad \eta = \eta_1, \qquad P = 0, \qquad \tau = \tau_1, \qquad \frac{X}{\lambda} = 1.$$

On the interval $\lambda_2 = \sqrt{\theta(e_\beta, \eta_1)} \leqslant \lambda \leqslant \lambda_1$, the solution is represented in parametric form:

$$\lambda = \sqrt{\theta(e, \eta_1)}, \qquad X = (1+e)\sqrt{\theta(e, \eta_1)} + \int_e^0 \sqrt{\theta(e, \eta_1)}\, de,$$

$$P = P(e, \eta_1), \qquad \tau = \tau(e, \eta_1) \qquad (e_\beta \leqslant e \leqslant 0).$$

On the interval $0 \leqslant \lambda \leqslant \lambda_2$:

$$P = \beta, \qquad e = e_\beta, \qquad \eta = \eta_1, \qquad \tau = \tau(e_\beta, \eta_1), \qquad X = (1+e_\beta)\lambda + \int_{e_\beta}^0 \sqrt{\theta(e, \eta_1)}\, de;$$

e_β is the abscissa of the point of intersection of the adiabat $\eta = \eta_1$ with the straight line $P = \beta$.

B. $P_2 < \beta = \beta_1 < P_1$. The picture of the motion differs from that obtained in 2°A in that the boundary between the nonstationary compression wave and the region of progressive motion will be a strong-discontinuity surface. The discontinuity surface corresponds to

$$\lambda_2^{*2} = \frac{\beta_1 - P_N}{e_M - e_N} = \theta(e_N, \eta_1) < \theta(0, \eta_1)$$

(MN is the tangent to the Poisson adiabat $\eta = \eta_1$, drawn from the point M).

C. If $P_3 < \beta = \beta_2 \leqslant P_2$, then there are two solutions satisfying all the conditions of the problem. One of them is constructed in the same way as in 2°B, the other as ···

in 1°. The choice of the unique solution can be made on the basis of considerations concerning the stability of strong-discontinuity surfaces. For the strong-discontinuity surface appearing in the second solution, $\theta(e_R, \eta_R) > \lambda^{*2} = \beta_2/e_R < \theta(0, \eta_1)$, and, consequently, the velocity relative to the medium behind and in front of the front is subsonic. Such a shock wave, as is known(4), is absolutely unstable.

D. For $\beta \leqslant P_3$ the solution is unique and is constructed as in 1°.

3°. Let us consider the case when, in the region $e < 0$, the derivative $(\partial^2 P/\partial e^2)_\eta$ changes sign twice (Fig. 3). The curve $B'K'$ is constructed for the arc $O'B'$ of the Poisson adiabat $\eta = \eta_2$, similarly to the curve BK in 2°.

Fig. 3

Figure 3: Fig. 3

Fig. 3

A. When $P_1 \leqslant \beta < 0$, and also when $\beta \leqslant P_5$, the picture of the motion is similar to that obtained in 1°; the solution is unique.

B. $P_3 \leqslant \beta < P_1$. When $P_2 \leqslant \beta_1 < P_1$, there are two solutions satisfying all the conditions of the problem. One of them is constructed as in 1°. However, since in the present case the strong-discontinuity surface is absolutely unstable $(\theta(e_R, \eta_R) < \lambda^{*2} > \theta(0, \eta_1))$[4], the solution under consideration must be rejected. The second solution, which occurs throughout the interval $P_3 \leqslant \beta < P_1$, corresponds to the following picture of the motion: ahead of the region of translational motion formed at the boundary of the medium, there propagates a nonstationary compression wave with a strong-discontinuity surface at its head ($\lambda_1^* = P_1/e_1 = \theta(e_1, \eta_2)$). The nonstationary compression wave is separated from the region of translational motion by a weak-discontinuity surface ($\lambda_2^2 = \theta(e_F, \eta_2)$).

C. $P_4 < \beta < P_3$. The boundaries of the nonstationary compression wave, separating it from the undisturbed medium and from the region of translational motion, become strong-discontinuity surfaces. For the leading shock wave $\lambda_1^{*2} = P_1/e_1 = \theta(e_1, \eta_2)$; for the one following it, λ_2^{*2} ($\lambda_2^* < \lambda_1^*$) is found similarly to λ_2^{*2} in 2°B. The solution is unique.

D. $P_5 < \beta \leqslant P_4$. There are two solutions satisfying all the conditions. One is constructed as in 3°C, the other as in 1°. The shock waves in each of these solutions are stable with respect to small disturbances of the type considered in[4].

If the shock adiabat Γ_1 with "center" $(0, 0)$ does not pass through the region G (or has a common point with the boundary LDL'—the point D), then outside the center $(\partial P/\partial e)_{\Gamma_1} > (\partial P/\partial e)_\eta$, the solution is unique and is constructed as in 1°. Cases in which the shock adiabat Γ_1 intersects the region G and has one common point with the adiabat $\eta = \eta_1$, or has no common points with it, differ only slightly from that analyzed in 3°.

We note that non-uniqueness of the solution also occurs in cases where $\partial\sigma/\partial S = 0$ and $\partial^2\sigma/\partial e^2$ changes sign, considered in[2], but it was not noticed by the author of the cited work.

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Note: Figure translations are in progress. See original paper for figures.