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Abstract

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GEODESIC FIELDS OF DIRECTIONS AND GROUPS OF HOMOTHETIES IN SPACES OF AFFINE CONNECTION

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I. The concept of a geodesic field of directions in A_n (an n-dimensional space of affine connection) was introduced by the author [1] and then by Hantzes [2]. Closely related to it is the concept of a torsion-forming field of directions, introduced by Yano [3].

Let in A_{n+1} ($n \geq 2$), referred to coordinates x^β, a field of directions be given by a vector field

$$A^\alpha(x^\beta); \qquad \alpha, \beta = 0, 1, \dots, n,$$

and let a two-dimensional surface (S) be formed by the trajectories ("A-lines") of the vector field A^α, intersecting a geodesic line of A_{n+1}.

If the surface S, constructed for any geodesic, is completely geodesic, then the field of directions determined by A^α will be called **geodesic**.

The necessary and sufficient conditions for the field of directions represented by A^α to be geodesic consist in the equalities [1]

$$A^\alpha{}_{,\beta} = T\delta^\alpha_\beta + B_\beta A^\alpha, \tag{1a}$$

$$R_{\mu(\beta\sigma)}{}^\alpha A^\mu = H^\alpha{}_{(\beta\sigma)} + F_{\beta\sigma} A^\alpha, \tag{1b}$$

where T is a certain scalar function; $B_\beta, H^\alpha{}_\beta$, and $F_{\beta\sigma}$ are certain tensors; $A^\alpha{}_{,\beta}$ denotes the covariant derivative of A^α, and parentheses denote symmetrization over the indices contained in them.

If only (1a) holds, then the field of directions is called **torsion-forming**.

In a space of constant curvature, any geodesic field of directions is formed by the directions of the straight lines of any connection. A more general example of spaces containing geodesic fields of directions is given by subprojective spaces.

Closely connected with a geodesic field of directions is the concept of a system of paths included in A_n. If, in some coordinate system u^α $(\alpha = 0, 1, \dots, n)$, $n-1$ equations of the system of geodesics of A_{n+1} can be written in the form

$$f_B(C_\theta, u^i) = 0; \qquad B = 1, \dots, n-1; \qquad i = 1, \dots, n, \tag{2}$$

containing $2(n-1)$ parameters C_θ in such a way that the latter are uniquely determined by the equalities

$$f_B(C_\theta, u_1^i) = 0, \qquad f_B(C_\theta, u_2^i) = 0,$$

then A_{n+1} includes the system of paths (2).

A subprojective space (A_{n+1} or V_{n+1}) may be characterized as a space containing a system of straight lines of an n-dimensional Euclidean space.

It has been proved that a system of paths included in A_{n+1} is a system of geodesics of some A_n, and that a necessary and sufficient condition for the inclusion in A_{n+1} of an n-dimensional system of paths is the presence in it of a geodesic field of directions; the included system of paths is isomorphic to the set of lines obtained by projecting—by means of the trajectories of the geodesic field of directions—the geodesic lines of A_{n+1} onto some one of its hypersurfaces ([4]).

II. Let A_{m+n} contain m geodesic fields of directions, represented by the vectors $\overset{a}{A}{}^\alpha$, satisfying (1a), (1b):

$$\overset{a}{A}{}^\alpha{}_{,\beta} = \overset{a}{T}\,\delta^\alpha_\beta + \overset{a}{B}_\beta \overset{a}{A}{}^\alpha, \tag{2a}$$

$$R_{\mu(\beta\sigma)}{}^\alpha \overset{a}{A}{}^\mu = \overset{a}{H}_{(\beta\sigma)}{}^\alpha + \overset{a}{F}_{\beta\sigma} \overset{a}{A}{}^\alpha. \tag{2 }$$

In these equalities and below, the indices from a to h take the values $1, \dots, m$, the indices from i onward take the values $m+1, \dots, m+n$, and the indices of the Greek alphabet take all the values $1, \dots, m+n$.

As follows from equality (2a), for the commutator vector

$$(\overset{a}{A}, \overset{b}{A})^\alpha = \overset{a}{A}{}^\alpha{}_{,\beta} \overset{b}{A}{}^\beta - \overset{b}{A}{}^\alpha{}_{,\beta} \overset{a}{A}{}^\beta$$

we have

$$(\overset{a}{A}, \overset{b}{A})^\alpha = \left(\overset{a}{T} - \overset{a}{B}_\beta \overset{b}{A}{}^\beta\right) \overset{b}{A}{}^\alpha - \left(\overset{b}{T} - \overset{b}{B}_\beta \overset{a}{A}{}^\beta\right) \overset{a}{A}{}^\alpha. \tag{I}$$

Hence follows the existence of coordinate systems (forming, we shall say, the class Σ') for which the trajectories of the vector fields $\overset{a}{A}$ are the coordinate u^a-lines.

Theorem 1. *If A_{n+m} contains m geodesic fields of directions, then in coordinates u^α of the class Σ' for its connection coefficients we have:*

$$\Gamma^\sigma_{\alpha\beta} = \Pi^\sigma_{\alpha\beta}(u^i) + \psi_{(\alpha\delta\beta)},$$

$$\Pi^b_{\alpha\beta} = \Pi^b_{\alpha\beta}(u^b, u^i), \qquad \Pi^i_{\alpha\beta} = \Pi^i_{\alpha\beta}(u^k), \tag{3}$$

$$\Pi^{P_a}_{a\beta} = 0, \qquad \Pi^{P_{a_1}}_{\alpha\beta} = 0,$$

where ψ_α are certain functions, $\Pi^b_{\alpha\beta}$ and $\Pi^i_{\alpha\beta}$ are certain functions of the indicated arguments, and the index P_a takes all values $1, \dots, m+n$, except one, namely a. Conversely, from conditions (3) it follows that the coordinate u^a-lines are trajectories of geodesic fields of directions.

As follows from (I), A_{n+m}, containing m geodesic fields of directions, is decomposed into ∞^n subspaces of m dimensions carrying the trajectories of these fields.

In a special coordinate system of the class Σ', these subspaces coincide with coordinate manifolds on which only the coordinates u^a vary; they turn out to be completely geodesic manifolds; moreover, each of them is mapped onto an m-dimensional Euclidean space in such a way that geodesics go over into straight lines. In connection with what has been said, there exist such coordinates u^α of the class Σ' that, in the equations of a geodesic belonging to the manifold $u^i = \text{const}$, the current coordinates

u^α are connected by linear relations; we shall call the class of such coordinate systems Σ''. For coordinates u^α of class Σ'', of course, (3) holds; moreover, one may set (by normalizing the quantities ψ_a in the corresponding way)

$$\Pi^a_{bc} = 0. \tag{3a}$$

Conversely, if (3), (3a) hold, then the coordinate system is of class Σ''.

Let ${}^aA^\alpha$ be m vectors ($m > 1$), representing m geodesic fields of directions in A_{n+m}, and suppose it is possible, by normalization—by multiplying ${}^aA^\alpha$ by certain functional factors—to achieve that: 1) for any constant values λ_a, the vector $\sum_a \lambda_a {}^aA^\alpha$ determines a geodesic field of directions; 2) among the vectors ${}^aA^\alpha$ there is no linear dependence with constant coefficients. The set of geodesic fields of directions determined in this case by the vectors $\sum_a \lambda_a {}^aA^\alpha$ will be called a **linear manifold of geodesic fields of directions of dimension** $m-1$.

The vector fields ${}^aA^\alpha$, normalized in the described manner, will be called ***L*-normalized**.

Theorem 2. In order that the vectors ${}^aA^\alpha$ satisfying (2a) and (2b) be L-normalized, it is necessary and sufficient that the equalities

$$ {}^1B_\beta = \dots = {}^mB_\beta; \qquad {}^1F_{\alpha\beta} = \dots = {}^mF_{\alpha\beta}. $$

hold.

Theorem 3. In order that the geodesic fields of directions determined by ${}^aA^\alpha$ belong to a linear manifold of similar fields, it is necessary and sufficient that: 1) the difference of any pair of vectors ${}^aB_\beta$ from (2a) be a vector-gradient; 2) ${}^1F_{\alpha\beta} = {}^mF_{\alpha\beta}$.

Theorem 4. If A_{n+m} admits a linear manifold of geodesic fields of directions of dimension $m-1$, then there exist coordinates u^α of class Σ'', for which the vectors ${}^aA^\alpha = \delta^\alpha_a$ (representing m geodesic fields of directions) are L-normalized.

We shall call them (i.e. the coordinates u^α) **coordinates of class** Σ.

Theorem 5. The connection coefficients of A_{n+m} with a linear manifold of geodesic fields of directions of dimension $m-1$ in a coordinate system (u^α) of class Σ have the form

$$ \Gamma^\alpha_{\beta\sigma} = \Pi^\alpha_{\beta\sigma} + \psi_{(\beta}\delta^\alpha_{\sigma)}, $$

$$ \Pi^{P_a}_{a\beta} = 0, \quad \Pi^a_{bc} = 0, \quad \Pi^i_{jk} = \Pi^i_{jk}(u^l), \quad \Pi^a_{ai} = \Pi_i(u^k), \tag{4} $$

$$ \Pi^a_{ij} = u^a C_{ij}(u^k) + {}^aD_{ij}(u^k), $$

where φ_β are certain functions; $\Pi^i_{jk}, \Pi_i, C_{ij}, {}^aD_{ij}$ are certain functions of the indicated arguments; the index P_a runs through all values $1, \dots, m+n$, except a.

The converse is also valid.

III. Let ${}^aA^\alpha \dfrac{\partial f}{\partial x^\alpha}$ be the operators of a group (of order m) of automorphic transformations of A_{n+m}.

If the field of directions determined by the vector $\sum_a \lambda_a A^a$ of any one-parameter subgroup is geodesic, then we shall call the transformations of the group **homotheties of the space** A_{n+m}.

The maximal order of the group of homotheties of A_n is realized in an affine space, for example; in this case the transformations reduce to homotheties in the usual sense, which in Cartesian coordinates are given by

$$x'^i = bx^i + a^i \qquad (i = 1, \dots, n),$$

where b, a^i are parameters.

Theorem 6. *The necessary and sufficient conditions that the vectors A^a ($a = 1, \dots, m$) determine in A_n a group of homotheties of order m consist in the equalities*

$$A^a_{\alpha,\beta} = T^a \delta^\alpha_\beta + B_\beta A^{a\alpha},$$

$$A^a_{\alpha,\beta\sigma} = A^{a\lambda} R_{\lambda\sigma\beta}{}^{\alpha},$$

where no restrictions are imposed on T^a, B_β (not following from the written equalities).

Theorem 7. *The group of homotheties of order m of the space A_n is locally isomorphic to the group of homotheties of an $(m-1)$-dimensional affine space.*

Theorem 8. *Geodesic fields of directions determined by vectors A^a that generate in A_n a group of homotheties of order m belong to an $(m-1)$-dimensional linear manifold of such fields.*

Theorem 9. *The systems of paths included in A_n ([3]) by means of the vectors of the group of homotheties are isomorphic.*

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Note: Figure translations are in progress. See original paper for figures.