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Abstract

Full Text

AERODYNAMICS

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ON THE STABILITY OF AN ELASTIC PLATE IN SUPERSONIC FLOW

(Presented by Academician L. I. Sedov, 13 I 1958)

On the basis of the linearized theory (i.e., under the assumptions that the disturbances of the incident flow, as well as the amplitudes of oscillation of the elastic plate, are sufficiently small), the problem of the stability of a plate of width a and small thickness h, moving translationally with supersonic velocity u, is reduced to consideration of an integro-differential equation for the deflection function $\zeta(x)$ of the following form:

$$\frac{d^4\zeta}{dx^4} + n\frac{d^2\zeta}{dx^2} - k^2\nu^2\zeta(x) = \lambda\left(\frac{d}{dx} + i\nu\right)\int_0^x K[x-\xi]\left\{\frac{d\zeta}{d\xi} + i\nu\zeta(\xi)\right\} d\xi, \quad (1)$$

where

$$K[x-\xi] = \exp\left[-\frac{i\nu M^2}{M^2-1}(x-\xi)\right] J_0\left[\frac{\nu M}{M^2-1}(x-\xi)\right],$$

$$\lambda = \frac{2\rho_a u^2 a^3}{D}\frac{1}{\sqrt{M^2-1}}, \qquad k = \sqrt{12}\, M\frac{a}{h}\left[c\Big/\sqrt{\frac{E}{\rho_m(1-\sigma^2)}}\right], \qquad n = \frac{Na^2}{D}.$$

Here x is the dimensionless coordinate in the direction of the flow $(0 \leq x \leq 1)$; $i\nu = \omega a/u = -\nu_1 + i\nu_2$ is the Strouhal number $(\nu_1, \nu_2$ real); ω is the complex frequency of the natural oscillations of the plate in the flow; N is the longitudinal compressive force; $D = Eh^3/12(1-\sigma^2)$ is the stiffness of the plate; ρ_m is the density of the plate material; ρ_a is the density of the undisturbed medium; $M = u/c$ is the Mach number.

The aerodynamic pressure on the surface of the plate is obtained from the Cauchy-Lagrange equation for the case of oscillating wings of infinite span, with allowance for the condition of compatibility of the oscillations of the elastic plate and the gaseous medium immediately near its surface.

Let us denote the aerodynamic action by

$$P\{\zeta(x)\} = \left(\frac{d}{dx} + i\nu\right) \int_0^x K[x-\xi] \left\{\frac{d\zeta}{d\xi} + i\nu\zeta(\xi)\right\} d\xi \tag{2}$$

and represent $\zeta(x)$ in the form of the sum $\zeta(x) = \zeta_1(x) + \zeta_2(x)$, where

$$\zeta_1(x) = \sum_{j=1}^{4} \omega_j e^{\beta_j x} = \sum_{j=1}^{4} \omega_j z_1^{(j)} \qquad (\omega_j \text{ are constants}),$$

$$\zeta_2(x) = \lambda \int_0^x G(x-\xi) P\{\zeta(\xi)\}\, d\xi,$$

$$\beta_{1,2} = \pm\beta = \pm\sqrt{\frac{n}{2}\left\{\left[1 + \left(\frac{2k\nu}{n}\right)^2\right]^{1/2} - 1\right\}},$$

$$\beta_{3,4} = \pm i\gamma = \pm i\sqrt{\frac{n}{2}\left\{\left[1 + \left(\frac{2k\nu}{n}\right)^2\right]^{1/2} + 1\right\}},$$

$$G(x-\xi) = \frac{1}{2(\beta^2+\gamma^2)} \left\{\frac{1}{\beta} e^{\beta(x-\xi)} - \frac{1}{\beta} e^{-\beta(x-\xi)} - \frac{1}{i\gamma} e^{i\gamma(x-\xi)} + \frac{1}{i\gamma} e^{-i\gamma(x-\xi)}\right\}.$$

Substituting $\zeta(x)$ into expression (1), we obtain the equation for the aerodynamic action

$$P\{\zeta(x)\} - \lambda \left(\frac{d}{dx} + i\nu\right) \int_0^x H(x-\xi)\, P\{\zeta(\xi)\}\, d\xi = f(x). \tag{3}$$

Here

$$H(x) = \int_0^x K[x-\xi] \left\{\frac{dG}{d\xi} + i\nu G(\xi)\right\} d\xi; \qquad f(x) = \sum_{j=1}^{4} \omega_j f^{(j)}(x),$$

$$f^{(j)}(x) = \left(\frac{d}{dx} + i\nu\right) \int_0^x K[x-\xi] \left\{\frac{d\zeta_1^{(j)}}{d\xi} + i\nu\zeta_1^{(j)}\right\} d\xi.$$

Let $P\{\zeta^{(j)}(x)\} = P^{(j)}(x)$ be the solution of equation (3) when its right-hand side is equal to $f^{(j)}(x)$. Denoting

$$\zeta_2^{(j)}(x) = \lambda \int_0^x G(x-\xi) P^{(j)}(\xi)\, d\xi,$$

we represent the solution in the form

$$\zeta(x) = \sum_{j=1}^{4} \omega_j \zeta^{(j)}(x), \qquad \zeta^{(j)}(x) = \zeta_1^{(j)}(x) + \lambda \int_0^x G(x-\xi) P^{(j)}(\xi)\, d\xi.$$

Then from equation (3) we obtain four independent equations

$$P^{(j)}(x) - \lambda \left(\frac{d}{dx} + i\nu \right) \int_0^x H(x-\xi) P^{(j)}(\xi)\, d\xi = f^{(j)}(x).$$

Denote

$$\widetilde{F}(s) = \int_0^\infty F(x) e^{-sx}\, dx, \qquad F(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \widetilde{F}(s) e^{sx}\, ds.$$

After transformation, the equation for $P^{(j)}(x)$ takes the form

$$\widetilde{P}^{(j)}(s) = \frac{(\beta_j + i\nu)(s + i\nu)\widetilde{K}(s)}{(s - \beta_j)\left[1 - \lambda(s + i\nu)^2 \widetilde{K}(s)\widetilde{G}(s)\right]},$$

$$\widetilde{K}(s) = \left[\left(s + i\frac{\nu M}{M+1} \right) \left(s + i\frac{\nu M}{M-1} \right) \right]^{-1/2}, \qquad \widetilde{G}(s) = \left(s^4 + ns^2 - k^2\nu^2\right)^{-1}.$$

By inverse transformation we obtain

$$P^{(j)}(x) = (\beta_j + i\nu) \int_0^x A(x-\xi) \zeta_1^{(j)}(\xi)\, d\xi,$$

$$A(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{(s + i\nu)(s^4 + ns^2 - k^2\nu^2) e^{sx}\, ds}{\sqrt{\left(s + \frac{i\nu M}{M+1}\right)\left(s + \frac{i\nu M}{M-1}\right)}\,(s^4 + ns^2 - k^2\nu^2) - \lambda(s + i\nu)^2}.$$

Similarly, we have

$$\zeta_2^{(j)}(x) = \lambda(\beta_j + i\nu) \int_0^x B(x-\xi) \zeta_1^{(j)}(\xi)\, d\xi;$$

where

$$B(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{(s+i\nu)\widetilde{K}(s)\widetilde{G}(s)e^{sx}}{1-\lambda(s+i\nu)^2\widetilde{K}(s)\widetilde{G}(s)}\,ds. \tag{4}$$

Now the question reduces to finding the zeros of the expression

$$\psi(s)=\sqrt{s^2+2i\nu s\frac{M^2}{M^2-1}-\nu^2\frac{M^2}{M^2-1}(s^4+ns^2-k^2\nu^2)}-\lambda(s+i\nu)^2. \tag{5}$$

We choose an integration contour consisting of two closed curves. The first curve consists of a part of a circle with center at the origin and the segment of the straight line $\mathrm{Re}(s)=\gamma$, situated to the right of all singularities and branch points. The second encloses the segment joining the branch points $-i\nu M/(M+1)$ and $-i\nu M/(M-1)$. It is evident that Jordan' s lemma is applicable here, since in (4) the expression under the integral sign before e^{sx} is of order $s^{-7/2}$.

We denote along the cut $i\nu=\omega,\ s+\dfrac{M}{M-1}\omega=\rho\omega$. In the case when, on the chosen branch, $\psi(s)$ has k simple zeros, the residue theorem gives

$$B(x)=-\int_0^{\frac{2M}{M^2-1}}g(\rho,\omega)\exp\left[x\omega\left(\rho-\frac{M}{M-1}\right)\right]d\rho+\sum_{l=1,2}^{k}\frac{s_l+i\nu}{[d\psi/ds]_{s=s_l}}e^{s_l x}, \tag{6}$$

$$g(\rho,\omega)=\frac{2\omega\left(\rho-\dfrac{1}{M-1}\right)\left|\sqrt{\rho\left(\dfrac{2M}{M^2-1}-\rho\right)}\right|G(\rho,\omega)}{\omega^2\rho\left(\dfrac{2M}{M^2-1}-\rho\right)G^2(\rho\omega)-\lambda^2\left(\rho-\dfrac{1}{M-1}\right)^4},$$

$$G(\rho,\omega)=\left(\rho-\frac{M}{M-1}\right)^4\omega^2+n\left(\rho-\frac{M}{M-1}\right)^2+k^2.$$

Consequently, the general solution (1) is represented in closed form as

$$\zeta(x)=\sum_{i=1}^{4}\omega_j\zeta^{(j)}(x)=\sum_{j=1}^{4}\omega_j\left[e^{\beta_j x}+\lambda(\beta_j+i\nu)\int_0^x B(x-\xi)e^{\beta_j\xi}d\xi\right] \tag{7}$$

or

$$\zeta(x) = \sum_{j=1}^{4} \omega_j \lambda(\beta_j + \omega) \left[\sum_{l=1}^{k} b_{jl} e^{s_{l x}} - \int_0^{\frac{2M}{M^2-1}} \frac{g(\rho,\omega)}{\omega\left(\rho - \frac{M}{M-1}\right) - \beta_j} \times \exp\left\{ x\omega\left(\rho - \frac{M}{M-1}\right)\right\} d\rho \right],$$

where the aerodynamic pressure is given by the expression

$$P(x) = \sum_{j=1}^{4} \omega_j (\beta_j + \omega) \left[\sum_{l=1}^{k} a_{jl} e^{s_{l x}} + a_{je}^{\beta_{jx}} - \int_0^{\frac{2M}{M^2-1}} \frac{\omega^2 G(\rho,\omega)}{\omega\left(\rho - \frac{M}{M-1}\right) - \beta_j} g(\rho,\omega) \exp\left[x\omega\left(\rho - \frac{M}{M-1}\right)\right] d\rho \right],$$

$$a_j = \frac{\beta_j + \omega}{\psi(\beta_j)} \frac{1}{\widetilde{G}(\beta_j)}, \qquad a_{jl} = b_{jl} \widetilde{G}(s_l) = \frac{s_l + \omega}{s_l - \beta_j} \frac{\widetilde{G}(s_l)}{[d\psi/ds]_{s=s_l}}.$$

Determination of the constants ω_j $(j = 1, 2, 3, 4)$ from the boundary conditions leads to the characteristic equation
$\chi[\nu, \lambda, k, n, M] = 0$, relating the frequency ν to the parameters of the problem.

For example, for a plate clamped along two edges,

$$\chi = \left| \begin{matrix} 2\beta\zeta^{(3)} - (\beta + i\gamma)\zeta^{(1)} - (\beta - i\gamma)\zeta^{(2)}(x) & 2\beta\zeta^{(4)} - (\beta - i\gamma)\zeta^{(1)} & -(\beta + i\gamma)\zeta^{(2)}(x) \\ 2\beta\zeta^{(3)'} - (\beta + i\gamma)\zeta^{(1)'} - (\beta - i\gamma)\zeta^{(2)'}(x) & 2\beta\zeta^{(4)'} - (\beta - i\gamma)\zeta^{(1)'} & -(\beta + i\gamma)\zeta^{(2)'}(x) \end{matrix} \right|_{x=1}. \tag{8}$$

Calculations with determinant (8) by means of the iteration method are apparently simpler than with the determinant of infinite order to which the problem under consideration is reduced when Galerkin’ s method is applied (2).

Let us note that the integrand in (6) is bounded for $-\pi/2 < \arg\omega < +\pi/2$, when the motion is stable, and the integral itself will be arbitrarily small for sufficiently large $M^2 \gg 1$. In the case of instability, the smallness of the integral will be ensured for low-frequency oscillations. For $M^2 \gg 1$, when

$$\widetilde{K}(s) = \left[s^2 + 2i\gamma s \frac{M^2}{M^2-1} - \gamma^2 \frac{M^2}{M^2-1} \right]^{-1/2} \approx \frac{1}{s + i\nu},$$

we represent the kernel in the simpler form $K[x - \xi] = e^{-i\nu(x-\xi)}$. Then solution (7) takes the form

$$\zeta(x) = \sum_{j=1}^{4} \omega_j \zeta^{(j)}(x), \qquad \zeta^{(j)}(x) = \lambda \sum_{l=1}^{k} \frac{\beta_j + i\nu}{s_l - \beta_j} \frac{e^{s_l x}}{[d\psi/ds]_{s=s_l}},$$

where s_l are the four roots of the equation

$$\psi(s) = s^4 + ns^2 - \lambda s - (i\gamma\lambda + k^2\nu^2) = 0.$$

It is not difficult to show that for $M^2 \gg 1$, in the cases of low-frequency oscillations, the aerodynamic pressure is given by an expression coinciding with the linear piston theory $P = \lambda[\partial\zeta/\partial x + i\nu\zeta]$.

Remark 1. In a number of cases, the problem of the stability of thin elastic bodies in supersonic flow is reduced to the investigation of an integro-differential equation of the indicated type. For example, in studying the influence of walls at a distance $2z_0$ on the oscillations of a plate located between them, in equation (5)

$$\widetilde{K}(s) = \left[s^2 + 2i\gamma s\frac{M^2}{M^2-1} - \gamma^2\frac{M^2}{M^2-1}\right]^{-1/2} \coth\left\{z_0\sqrt{s^2 + 2i\gamma s\frac{M^2}{M^2-1} - \gamma^2\frac{M^2}{M^2-1}}\right\}.$$

Under certain assumptions, the problem of the stability of wings of finite span in supersonic flow is reduced to a system of equations of a similar form, for the investigation of which the methods considered above may be applied.

Remark 2. The method of solution using the aerodynamic-pressure function is applicable to the investigation of oscillations of a multi-span plate with supports of arbitrary type, perpendicular to the direction of the flow and situated at arbitrary distances from one another. In this case, the equation of oscillations for each section has a nonhomogeneous term depending on the aerodynamic pressure of the preceding sections. The problem of self-oscillations of a single-span plate of infinite span is also solved by direct application of the Laplace transform to the original equation.

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