ON THE FLOW OF A CONDUCTING LIQUID PAST MAGNETIZED BODIES

1957

SovietRxiv

Abstract

Full Text

HYDROMECHANICS

A. G. KULIKOVSKII

ON THE FLOW OF A CONDUCTING LIQUID PAST MAGNETIZED BODIES

(Presented by Academician L. I. Sedov, 17 VI 1957)

If the conductivity of the liquid is infinite, then the oncoming flow, in which the magnetic field is assumed to be absent, cannot penetrate into the region occupied by the magnetic field (for infinite conductivity the magnetic lines of force are "frozen" into the liquid). It follows from this that the region being flowed around consists of a magnetized body and a "cavity" —a region in which the magnetic field is located. This region may be empty and may be filled with liquid. For definiteness we shall assume that the cavity is filled with the flowing liquid, since for large but finite conductivity the liquid, although slowly, can penetrate into the cavity.

Let us consider the case of steady flow. Inside the cavity, generally speaking, an established motion of the liquid is possible, but in what follows we shall assume that the liquid is at rest inside the cavity, and we shall seek the shape of the cavity and the magnetic field in it. We note that the velocity in the cavity is necessarily equal to zero when all magnetic lines of force begin and end on the body. Putting $v = 0$ in the equations of magnetohydrodynamics, we obtain:

$$\operatorname{div}\mathbf{H} = 0, \qquad \operatorname{grad} p_{\text{cv}} = \frac{1}{4\pi}[\operatorname{rot}\mathbf{H}\cdot\mathbf{H}], \tag{1}$$

where p_{cv} is the pressure in the cavity; moreover, on the boundary of the cavity with the flow the conditions must be satisfied (p_{fl} is the pressure in the flow, $\mathbf{n}$ is the unit vector of the normal)

$$(\mathbf{H}\cdot\mathbf{n}) = 0, \qquad p_{\text{fl}} = p_{\text{cv}} + \frac{1}{8\pi}H^2 \tag{2}$$

The first condition follows from the absence of sources of the magnetic field and is satisfied on the entire boundary of the region being flowed around, while the second expresses the fact of equality of the normal stresses (the tangential magnetic stress is automatically equal to zero when the first condition (2) is fulfilled).

Inside the body a magnetic field may be prescribed (for example, when the conductivity of the bodies is infinite), or currents may be prescribed in the body

$$\mathbf{j}_{\mathrm{t}} = \frac{c}{4\pi} \operatorname{rot} \mathbf{H}, \tag{3}$$

and the intensity of the magnetic field in the body must be determined from the solution of the problem. In this case, on the boundary of the body with the cavity, the normal components of the vectors $\mathbf{H}$ and $\operatorname{rot}\mathbf{H}$ must be continuous along the normal.

But equations (1), with their boundary conditions and the conditions inside the body, do not give a unique solution of the problem. Indeed, sta-

stationary motion, regarded as having arisen from a state of rest, may depend on how the magnetic field was "frozen into" the fluid in the initial state.

In order that the solution of the problem should not depend on the initial state, but only on the conditions inside the body, let us impose on the magnetic field the following additional conditions (which, generally speaking, are not necessary).

Let, inside the cavity,

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = 0, \tag{4}$$

and on the boundary of the cavity with the flow

$$[\operatorname{rot}\mathbf{H} \cdot \mathbf{n}] = 0. \tag{5}$$

These conditions were obtained in an attempt to single out the solution to which stationary solutions with finite constant conductivity σ tend as $\sigma \to \infty$. It was assumed here that in the neighborhood of the body there is a region where $\mathbf{v} \to 0$, and also that in the generalized Ohm law

$$\mathbf{j} = \sigma \left\{ \mathbf{E} + \frac{1}{c} [\mathbf{v} \cdot \mathbf{H}] \right\}$$

the last term tends to zero as $\sigma \to \infty$.

Equations (4) and (3) can be combined into a single equation describing the magnetic field in the entire region being flowed around:

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = \frac{4\pi}{c} \operatorname{rot}\mathbf{j}_T, \tag{6}$$

where $\mathbf{j}_T$ is a function coinciding with the currents inside the body and equal to zero outside the body; moreover, on the boundary of the body with the flow, if such a boundary exists, the vector $\mathbf{j}_T$ itself must be specified.

We now show that if the shape of the cavity is known in advance and the region being flowed around is finite, then equation (6), together with the first equation (1) and its boundary conditions, can have only a unique solution. For this purpose we show that the difference of two solutions $\mathbf{H}_1 - \mathbf{H}_2 = \mathbf{H}_3$ is equal to zero. Indeed, $\mathbf{H}_3$ everywhere in the region being flowed around satisfies equation (4), and on its entire boundary satisfies condition (5) and the first condition (2). Hence we conclude that

$$\operatorname{rot} \mathbf{H}_3 = \operatorname{grad} f \tag{7}$$

and that on the boundary $f = \text{const}$. Taking div of both sides of equality (7), we obtain $\Delta f = 0$, and, consequently, everywhere in the region being flowed around $\operatorname{grad} f = 0$. Then from equality (7) it follows that $\mathbf{H}_3 = \operatorname{grad} \psi$, and, taking into account the first equality (1), we obtain $\Delta\psi = 0$, while on the boundary, by virtue of the first condition (2), the equality $\partial\psi/\partial n = 0$ will hold. Consequently, $\psi = \text{const}$ and $\mathbf{H}_3 = 0$.

Equation (6) is simplified when the problem is plane or axisymmetric. In the plane case let us introduce coordinates x, y, z (the axes x and y lie in the plane of symmetry), and in the axisymmetric case coordinates x, y, φ (the x-axis coincides with the axis of symmetry). In both cases the currents inside the cavity

$$\mathbf{j} = \frac{c}{4\pi} \operatorname{rot} \mathbf{H}$$

lie in the xy-plane. This follows from equalities (4) and (5), and also from the fact that in the incident flow $\mathbf{H} = 0$. Therefore equation (6) can be written in the following form:

in the plane case

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{4\pi}{c} j_{Tz},$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} = \frac{4\pi}{c} \left(\frac{\partial j_{Tx}}{\partial y} - \frac{\partial j_{Ty}}{\partial x} \right);$$

in the axisymmetric case

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \frac{4\pi}{c} j_{t\varphi},$$

$$\frac{\partial^2 H_\varphi}{\partial x^2} + \frac{\partial}{\partial y} \left[\frac{1}{y} \frac{\partial}{\partial y} (y H_\varphi) \right] = \frac{4\pi}{c} \left(\frac{\partial j_{tx}}{\partial y} - \frac{\partial j_{ty}}{\partial x} \right),$$

where condition (5) will have the form:

in the plane case

$$\frac{\partial H_z}{\partial n} = 0,$$

in the axisymmetric case

$$\frac{\partial H_\varphi}{\partial n} + \frac{H_\varphi}{y} n_y = 0,$$

where $\mathbf{n} = in_x + jn_y$ is the unit normal vector. If $\partial j_{tx}/\partial y - \partial j_{ty}/\partial x = 0$, then H_z and H_φ may be equal to zero, and in this case the problem reduces to integrating the two equations

$$\operatorname{rot} \mathbf{H} = \mathbf{j}_t, \qquad \operatorname{div} \mathbf{H} = 0,$$

and from the second equation (1) we obtain $p_{\text{kv}} = \text{const}$.

Let us consider several simplest examples, assuming that $H_z = 0$ and $H_\varphi = 0$.

1. **Flow of an incompressible fluid past a plane magnetic dipole located perpendicular to the flow.** The surface being flowed around is a cylinder, whose radius we shall denote by a. The magnetic field inside the cylinder and the velocity of the fluid outside it are given by the formulas

$$\mathbf{H} = \operatorname{grad} H_0 y \left(1 - \frac{a^2}{x^2 + y^2}\right), \qquad \mathbf{v} = \operatorname{grad} U_0 x \left(1 + \frac{a^2}{x^2 + y^2}\right),$$

where U_0 and H_0 are constants related by

$$H_0^2 = 8\pi\rho U_0^2.$$

The pressure inside the cylinder p_k is constant and is related to the stagnation pressure of the incident flow p_0 as follows:

$$p_0 - p_k = \frac{1}{4\pi} H_0^2.$$

2. **Supersonic flow past a wedge, along the surface of which a current of constant density i flows parallel to the edge of the wedge.** For definiteness we shall assume that along the different faces of the wedge the current flows in opposite directions. The pattern of flow and of the behavior of the magnetic lines of force is shown in Fig. 1.

Fig. 1

Figure 1: Fig. 1

The straight line I represents the plane of symmetry, straight line II the shock wave, straight line III the cavity boundary, and straight line IV the wedge boundary. The streamlines after the shock wave run parallel to the cavity boundary; the pressure of the external flow p_n is constant. The magnetic lines of force in the cavity run parallel to its boundary, and the magnetic-field intensity

Fig. 1

H_{cav} is constant. The pressure in the cavity is related to the pressure in the flow by the relation

$$p_{\text{p}} = p_{\text{cav}} + \frac{1}{8\pi} H_{\text{cav}}^2.$$

Inside the wedge the magnetic-field strength H_{w} is constant and is related to H_{cav} and i by the relations

$$H_{\text{cav}} \sin\alpha = H_{\text{w}} \sin\beta, \qquad i = \frac{c}{4\pi}\left(H_{\text{cav}}\cos\alpha + H_{\text{w}}\cos\beta\right).$$

It is easy to see that the minimum current strength necessary to maintain the pressure difference $p_{\text{p}} - p_{\text{cav}}$ is attained when $\alpha = 0$ and is equal to

$$i_{\text{min}} = \frac{c}{\sqrt{2\pi}}\sqrt{p_{\text{p}} - p_{\text{cav}}}.$$

i_{min} is a very large quantity; thus, for example, if $p_{\text{p}} - p_{\text{cav}} = 1$ atm, then $i_{\text{min}} = 4000$ A/cm.

Thus, a necessary condition for the existence of a cavity is $i > i_{\text{min}}$. Otherwise the cavity is absent and ordinary flow past the wedge takes place.

3. Supersonic flow past a cone, over whose surface there flows a current of constant density i, whose direction is perpendicular to the generator of the cone. The flow pattern remains approximately the same, but the streamlines behind the shock wave and the magnetic lines of force in the cavity cease to be straight. The behavior of the magnetic lines of force in the cavity is given by the formulas

$$H_r = C_1\left(1 + \cos\theta \ln \operatorname{tg}\frac{\theta}{2}\right) + C_2\cos\theta,$$

$$H_\theta = C_1 \left(\operatorname{ctg}\theta - \sin\theta \ln \operatorname{tg}\frac{\theta}{2}\right) - C_1 \sin\theta.$$

Here H_r is the component of the magnetic-field strength along the radius vector drawn from the vertex of the cone; θ is the angle of the radius vector with the axis of symmetry; H_θ is the component of the magnetic-field strength perpendicular to H_r. The constants C_1 and C_2 are determined from the conditions at the boundary of the cavity (for $\theta = \alpha + \beta$)

$$H_\theta = 0, \qquad p_{\text{p}} = p_{\text{cav}} + \frac{1}{8\pi} H_r^2.$$

Inside the cone the magnetic-field strength is constant, and the magnetic lines of force are parallel to the axis of symmetry. The surface-current density i is determined from the equality

$$i = \frac{c}{4\pi} \left(H_{\text{cav}\,\tau} - H_{\text{cone}\,\tau}\right),$$

where $H_{\text{cav}\,\tau}$ and $H_{\text{cone}\,\tau}$ are the tangential components of the magnetic-field strengths of the cavity and the cone on the surface of the cone. For the cone there also exists an $i_{\min}$, which is expressed by the same formula as for the wedge.

Moscow State University
named after M. V. Lomonosov

Received
2 VI 1957

Note: Figure translations are in progress. See original paper for figures.