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Abstract

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ON THE RELATION BETWEEN COMPLETE AND PARTIAL BEST MEAN APPROXIMATIONS OF FUNCTIONS OF MANY VARIABLES

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Consider the space L_p $(1 \le p < \infty)$ of all measurable functions $f(x_1, \dots, x_k)$, of period 2π in each of the variables x_i $(i = 1, 2, \dots, k)$, whose p-th power of the modulus is integrable on the k-dimensional cube of periods, with norm

$$\|f\|_{L_p} = \left\{ \int_0^{2\pi} \cdots \int_0^{2\pi} |f(x_1, \dots, x_k)|^p \, dx_1 \cdots dx_k \right\}^{1/p}.$$

Let

$$E_{n_1,\dots,n_k}(f)_{L_p} = \inf_T \|f(x_1, \dots, x_k) - T_{n_1,\dots,n_k}(x_1, \dots, x_k)\|_{L_p}$$

be the complete best approximation of the function f by trigonometric polynomials of order $\le n_i$ in the variables x_i $(i = 1, 2, \dots, k)$.

By Fubini' s theorem, for any $r < k$ the function $f(x_1, x_2, \dots, x_k)$, as a function of the variables $x_1, \dots, x_r$, for almost all systems $(x_{r+1}, \dots, x_k)$, also belongs to the class L_p together with its best approximation $E_{n_1,\dots,n_r}(f; x_{r+1}, \dots, x_k)$ in the chosen r variables. The quantity

$$E_{n_1,\dots,n_r,\infty}(f) = \|E_{n_1,\dots,n_r}(f; x_{r+1}, \dots, x_k)\|_{L_p}$$

may be regarded as the partial best approximation of order n_i with respect to the variables x_i $(i = 1, 2, \dots, r)$. This quantity coincides with the lower bound

$$\inf \|f(x_1, \dots, x_k) - T_{n_1,\dots,n_r}[x_1, \dots, x_r; \, (x_{r+1}, \dots, x_k)]\|_{L_p}$$

over all possible trigonometric polynomials of order n_i in the variables x_i $(i = 1, 2, \dots, r)$, whose coefficients are periodic functions $\varphi_{i_1,\dots,i_r}(x_{r+1}, \dots, x_k)$ of period 2π in each of the variables x_i $(i = r + 1, \dots, k)$, belonging to L_p. Owing to this, the inequality

$$E_{n_1,\dots,n_k}(f) \geq E_{n_1,\dots,n_r,\infty}(f)$$

always holds.

The following theorem complements this estimate and indicates a closer connection between complete and partial approximations.

Theorem. For any finite $p > 1$ there exists a constant C_p, independent of the function f, such that

$$E_{n_1,\dots,n_k}(f)_{L_p} \leq C_p \min\left\{E_{n_{\nu_1},\dots,n_{\nu_i},\infty}(f)_{L_p} + E_{n_{\nu_{i+1}},\dots,n_{\nu_k},\infty}(f)_{L_p}\right\} \qquad (1)$$

$$(\nu_m = 1, 2, \dots, k; \quad m = 1, 2, \dots, i).$$

In the cases $p = 1$, $p = \infty$, the inequality

$$E_{n_1,\dots,n_k}(f) \leq C \min\{(E_{n_{\nu_1},\dots,n_{\nu_i},\infty}(f) + E_{n_{\nu_{i+1}},\dots,n_{\nu_k},\infty}(f)) \ln n_{\nu_1} \cdots \ln n_{\nu_i}\} \ (2)$$

$$\left(\nu_m = 1, 2, \dots, k; \quad m = 1, 2, \dots, i; \quad i \leq \left[\frac{k}{2}\right]\right),$$

where C is an absolute constant.

For continuous functions of two variables in the case of the uniform metric ($p = \infty$), inequality (2) was obtained by S. N. Bernstein (^{1}), who also indicated the special case of estimate (1), for $p = 2$ with constant $C_2 = 1$, that follows from Parseval' s equality.

We shall give the proof of inequalities (1) and (2) only for the case of functions of two variables.

Proof of inequality (1). Let $T_{n_1}[x_1; (x_2)]$, $T_{n_2}[(x_1); x_2]$ be trigonometric polynomials realizing the partial best approximations to the function $f(x_1, x_2)$, the first of order n_1 in x_1, the second of order n_2 in x_2, i.e.

$$E_{n_1,\infty}(f) = \|f(x_1, x_2) - T_{n_1}[x_1; (x_2)]\|, \qquad E_{n_2,\infty}(f) = \|f(x_1, x_2) - T_{n_2}[(x_1); x_2]\|.$$

Denote:

$$S_{n_1}(f; x_1, x_2) = \frac{1}{\pi} \int_0^{2\pi} f(x_1 + t_1, x_2) D_{n_1}(t_1)\, dt_1,$$

$$S_{n_1,n_2}(f;x_1,x_2)=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1+t_1,x_2+t_2)D_{n_1}(t_1)D_{n_2}(t_2)\,dt_1dt_2,$$

where

$$D_n(t)=\frac{\sin(2n+1)\frac{t}{2}}{2\sin\frac{t}{2}}.$$

It is obvious that

$$S_{n_1,n_2}(T_{n_1};x_1,x_2)=S_{n_2}(T_{n_1};x_1,x_2)=\frac{1}{\pi}\int_0^{2\pi} T_{n_1}[x_1,(x_2+t_2)]D_{n_2}(t_2)\,dt_2.$$

It follows from this that

$$E_{n_1,n_2}(f)\le\|f(x_1,x_2)-S_{n_2}(T_{n_1};x_1,x_2)\|\le\|f(x_1,x_2)-S_{n_2}(f;x_1,x_2)\|+$$
$$+\|S_{n_2}(f;x_1,x_2)-S_{n_2}(T_{n_1};x_1,x_2)\|=R_1+R_2. \qquad (3)$$

To estimate each term on the right-hand side of (3), we shall use Riesz' s inequality (^{2})

$$\|S_m(f)\|_{L_p}\le A_p\|f\|_{L_p} \qquad (p>1).$$

Then, obviously,

$$R_2\le A_p\|f(x_1,x_2)-T_{n_1}[x_1;(x_2)]\|=A_pE_{n_1,\infty}(f); \qquad (4)$$

$$R_1\le\|f(x_1,x_2)-T_{n_2}[(x_1);x_2]\|+\|T_{n_2}[(x_1);x_2]-S_{n_2}(f;x_1,x_2)\|=$$
$$=E_{n_2,\infty}(f)+\|S_{n_2}(f-T_{n_2};x_1,x_2)\|\le E_{n_2,\infty}(f)+A_pE_{n_2,\infty}(f). \qquad (5)$$

From (4) and (5), (1) follows.

Proof of inequality (2). Considering inequality (3) in the metric L, we estimate R_1 and R_2.

Changing the order of integration and, by virtue of the periodicity of the function in each variable, we obtain:

$$R_2 \leq \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\pi} \int_0^{2\pi} \Big|f(x_1, x_2 + t_2) - T_{n_1}[x_1, (x_2 + t_2)]\Big| \cdot |D_{n_2}(t_2)| \, dt_2 \, dx_1 \, dx_2 \leq$$

$$\leq \Big\|f(x_1, x_2) - T_{n_1}[x_1; (x_2)]\Big\| \cdot \frac{1}{\pi} \int_0^{2\pi} |D_{n_2}(t_2)| \, dt_2 =$$

$$= E_{n_1,\infty}(f) \cdot \frac{1}{\pi} \int_0^{2\pi} |D_{n_2}(t_2)| \, dt_2, \tag{6}$$

$$R_1 \leq \Big\|f(x_1, x_2) - T_{n_2}[(x_1); x_2]\Big\| + \Big\|T_{n_2}[(x_1); x_2] - S_{n_2}(f; x_1, x_2)\Big\| \leq$$

$$\leq E_{n_2,\infty}(f) + \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\pi} \int_0^{2\pi} \Big|f(x_1, x_2 + t_2) - T_{n_2}[(x_1); x_2 + t_2]\Big| \cdot |D_{n_2}(t_2)| \, dt_2 \, dx_1 \, dx_2 \leq$$

$$\leq E_{n_2,\infty}(f) + \frac{1}{\pi} \int_0^{2\pi} |D_{n_2}(t_2)| \, dt_2 \cdot E_{n_2,\infty}(f). \tag{7}$$

From (6) and (7) we obtain

$$E_{n_1,n_2}(f) \leq C\{E_{n_1,\infty}(f) + E_{n_2,\infty}(f)\} \cdot \ln n_2. \tag{8}$$

Analogously one can also obtain the inequality

$$E_{n_1,n_2}(f) \leq C\{E_{n_1,\infty}(f) + E_{n_2,\infty}(f)\} \cdot \ln n_1. \tag{9}$$

(8) and (9) give (2).

The same method makes it possible to prove inequalities (8) and (9) for the case of the uniform metric.

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Note: Figure translations are in progress. See original paper for figures.