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Abstract

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MATHEMATICS

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ON SIMULTANEOUS APPROXIMATION IN THE MEAN OF COMPLEX-VALUED FUNCTIONS GIVEN ON SEVERAL CONTOURS

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1. Let γ be a closed Jordan rectifiable curve. Denote by s the length of the arc of the curve γ, measured from some point $\zeta_0 \in \gamma$; $0 \leq s \leq l$, where l is the length of γ; let $\sigma(s)$ be a nondecreasing function of bounded variation for $0 \leq s \leq l^*$. Consider the space $L^p(d\sigma, \gamma)$, $p > 0$, of complex-valued functions $f(\zeta)$, defined on γ, for which

$$\int_\gamma |f(\zeta)|^p \, d\sigma(s) < \infty^{**}.$$

In our note ([2]) a theorem was given in which a characterization was supplied of the closure of the linear span of the system $\{\zeta^m\}$, $m = 0, 1, 2, ...$, in $L^p(d\sigma, \gamma)$, $p > 0$. In the present note we shall consider the approximation of functions given on several contours. Let G be an n-connected domain bounded by n closed rectifiable curves $\gamma_1, \dots, \gamma_n$. For definiteness we shall regard G as a finite domain, and γ_1 as the outer contour. The complete boundary of G will be denoted by Γ. For each of the curves γ_i, $i = 1, 2, \dots, n$, we define the spaces $L^p(d\sigma_i, \gamma_i)$ and shall henceforth denote by $L^p(d\sigma, \Gamma)$ the totality of all complex-valued functions $f(\zeta)$, defined on Γ, belonging on each component γ_i of the boundary Γ to the corresponding space $L^p(d\sigma_i, \gamma_i)$. The distance between two functions $f_1(\zeta)$ and $f_2(\zeta)$ belonging to $L^p(d\sigma, \Gamma)$ will be defined by the formula

$$\rho(f_1, f_2) = \int_\Gamma |f_1 - f_2| \, d\sigma = \sum_{i=1}^{n} \int_{\gamma_i} |f_1 - f_2| \, d\sigma_i.$$

2. We first consider the approximation of functions in the metric $L^p(d\sigma, \Gamma)$ by sequences $\{\Pi_k(\zeta)\}$ of polynomials in ζ. For this purpose we investigate the behavior, inside a simply connected domain g bounded by a curve γ, of a sequence of polynomials $\{\Pi_k(z)\}$ satisfying on γ the condition

$$\lim_{k\to\infty}\int_\gamma |f(\zeta)-\Pi_k(\zeta)|^p\,d\sigma=0, \qquad p>0. \tag{1}$$

Theorem 1. *If for $\sigma(s)$ the condition*

$$\int_\gamma \ln\sigma'(s)\,|\psi'(\zeta)\,d\zeta|>-\infty, \tag{2}$$

is fulfilled, where $w=\psi(z)$ maps g conformally onto $|w|<1$, then the sequence $\{\Pi_k(z)\}$, satisfying condition (1), will converge uniformly inside g

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* $\sigma(s-0)=\sigma(s)$.

** The integrals are understood in the Lebesgue-Stieltjes sense.

converge, and the function $f(z)=\lim_{k\to\infty}\Pi_k(z)$ will have on γ almost everywhere the angular boundary values $f(\zeta)$ coinciding with the function appearing in equality (1).

Theorem 2. If condition (2) is not fulfilled for $\sigma(s)$, then for any function $f(\zeta)\in L^p(d\sigma,\gamma)$, $p>0$, and an arbitrary function $F(z)$ analytic in the domain g, there exists a sequence $\{\Pi_k(z)\}$ such that on γ (1) will hold, while inside g $\{\Pi_k(z)\}$ will converge uniformly to $F(z)$.

Since condition (2) is necessary and sufficient for the non-closedness of $\{\zeta^m\}, m=0,1,\dots,$ in $L^p(d\sigma,\gamma)$*, Theorems 1 and 2 show that, in the case of non-closedness of the system $\{\zeta^m\}, m=0,1,2,\dots,$ in $L^p(d\sigma,\gamma)$, the convergence of $\{\Pi_k(\zeta)\}$ on γ in the metric of $L^p(d\sigma,\gamma)$ implies the uniform convergence of $\{\Pi_k(z)\}$ inside g, while in the case of closedness of the system under consideration the requirement that $\{\Pi_k(\zeta)\}$ converge on γ in the metric of $L^p(d\sigma,\gamma)$ "does not constrain" the behavior of $\{\Pi_k(z)\}$ in closed subdomains lying inside g. In the latter case, using Theorem 2, it is easy to construct examples of $\{\Pi_k(z)\}$ satisfying condition (1) and converging uniformly inside g to ∞, etc.

Remark. The dependence of the behavior inside the domain of polynomials whose norms in $L^p(d\sigma,\gamma)$ are uniformly bounded on whether the system $\{\zeta^m\}, m=0,1,\dots,$ is closed or non-closed was also pointed out by S. N. Mergelyan, who established that in the case of closedness of $\{\zeta^m\}, m=0,1,2,\dots,$ in $L^p(d\sigma,\gamma)$ the set of polynomials under consideration is noncompact inside g, while in the case of non-closedness it is compact.

3. With the aid of Theorems 1 and 2 and the earlier investigation of approximation by polynomials of functions defined on a closed rectifiable curve, one can completely study the question of approximation by sequences of

polynomials of functions defined on a composite contour Γ. Here we restrict ourselves only to formulating a theorem giving a sufficient condition for the closedness of the system $\{\zeta^m\}, m = 0, 1, 2, \ldots$.

Theorem 3. If the system $\{\zeta^m\}, m = 0, 1, 2, \ldots$, is closed in each of the spaces $L^p(d\sigma_i, \gamma_i)$, $i = 1, \ldots, n$, $p > 0$, then this system is closed also in the space $L^p(d\sigma, \Gamma)$.

Remark. Theorem 3 would be a simple consequence of Runge' s theorem only in the case when the finite domains bounded by the curves γ_i, $i = 1, \ldots, n$, had no common points. Recall that in our case $\gamma_2, \ldots, \gamma_n$ lie inside γ_1. Therefore the possibility of simultaneous approximation by one and the same sequence of polynomials of functions prescribed on $\gamma_1, \ldots, \gamma_n$ requires special consideration. We also note that Theorem 3 will be valid for any system of pairwise nonintersecting contours γ_i. The necessity of the conditions of Theorem 3 is obvious.

4. Let us now consider the question of which functions $f(\zeta)$, defined on Γ, can be approximated arbitrarily well in the metric $L^p(d\sigma, \Gamma)$ by sequences of boundary values of functions analytic in the closed domain $\overline{G}$. For this it is necessary to investigate the closure of the linear span of the system of functions

$$\left\{z^m, \ \frac{1}{(z-\alpha_i)^m}\right\}, \qquad m = 0, 1, 2, \ldots, \quad i = 1, \ldots, n, \tag{3}$$

where α_i is a point chosen inside the domain complementary to $\overline{G}$ and bounded by γ_i. As the investigation shows, the resulting

* For $p \geqslant 1$ this was first proved by Ya. L. Geronimus.

the results turn out to be analogous to those obtained in considering approximation by polynomials of functions defined on a closed Jordan rectifiable curve (1). In what follows we shall denote by $R(z)$ a linear combination of the functions of the system (3).

Theorem 4. For the system (3) to be closed in $L^p(d\sigma, \Gamma)$, $p > 0$, it is necessary and sufficient that

$$\int_\Gamma \ln \sigma'(s) \, |\Psi'(\zeta) \, d\zeta| = -\infty, \tag{4}$$

where $w = \Psi(z)$ maps the domain G onto the n-connected circular canonical domain K.*

Definition. An analytic function $F(z)$ in the domain G belongs to the class D if the subharmonic functions $\ln^+ \left|\frac{F(z)}{M}\right|$ have in the domain G least harmonic majorants $u^M(z)$ satisfying the condition $\lim_{M\to\infty} u^M(z_0) = 0$, where $z_0 \in G$. It follows from this definition that the class D is a subclass of the functions of bounded characteristic in G. In the case when the domain G is the unit disk, it is not difficult to see that the class D introduced above coincides with the class of functions satisfying the condition of P. Ya. Polubarinova-Kochina ((1), Ch. II, Sec. 6.5).

Theorem 5. If condition (4) is not satisfied, then in order that the function $f(\zeta) \in L^p(d\sigma, \Gamma)$, $p > 0$, belong to the closure of the linear span of the system (3) in the space under consideration, it is necessary and sufficient that there exist an analytic function $f(z)$ of class D in G, whose angular boundary values would coincide almost everywhere on Γ with $f(\zeta)$.

Remark. It can be proved that condition (4) is equivalent to the following: among the integrals

$$\int_{\gamma_i} \ln \sigma_i'(s)\, |\psi'(\zeta)\, d\zeta|, \quad i = 1, 2, \dots, n,$$

where $w = \psi_i(z)$, $i = 1, \dots, n$, maps conformally the domain G_i, bounded by γ_i, $G_i \supset G$, onto $|w| < 1$ for $i = 1$ and onto $|w| > 1$ for $i \neq 1$, at least one is equal to $-\infty$.

5. For multiply connected domains there will hold theorems analogous to Theorems 2 and 3.

Theorem 6. If $\{R_k(z)\}$ converges in the metric $L^p(d\sigma, \Gamma)$, $p > 0$, where $\sigma(s)$ does not satisfy condition (4), then inside G the sequence $\{R_k(z)\}$ will converge uniformly to that analytic function $f(z)$ to whose boundary values $\{R_k(\zeta)\}$ converged.

Theorem 7. If for $\sigma(s)$ condition (4) is satisfied, then for any preassigned functions $f(\zeta) \in L^p(d\sigma, \Gamma)$, $p > 0$, and $\Phi(z)$, analytic inside G, there exists $\{R_k(z)\}$ such that on the boundary Γ the sequence $\{R_k(\zeta)\}$ converges in the metric $L^p(d\sigma, \Gamma)$ to $f(\zeta)$, while inside G it converges uniformly to $\Phi(z)$.

6. We give a theorem that provides a sufficient condition for an analytic function $f(z)$ in the domain G to belong to the class D.

Theorem 8. If for $f(z)$ there exists a sequence of analytic functions $\{f_k(z)\}$, bounded in G, converging uniformly to $f(z)$ inside G, and such that

$$\int_{\Gamma} |f_k(\zeta)|^p \rho(\zeta)\, |d\zeta| \leq \infty, \quad k = 1, 2, \dots,$$

where $\rho(\zeta) \geq 0$ satisfies the condition

$$\int_{\Gamma} \ln \rho(\zeta)\, |\psi'(\zeta)\, d\zeta| > -\infty,$$

then $f(z) \in D$, and the boundary values $f(\zeta)$ are summable on Γ to the power p with weight $\rho(\zeta)$.

* Instead of a mapping onto the circular domain K, one may take mappings onto any n-connected domain bounded by analytic curves.

From Theorems 8, 5, and 6 it follows:

Corollary. Under the hypotheses of Theorem 8 there exists $\{R_k(z)\}$, converging uniformly inside G to $f(z)$, and such that

$$\lim_{k\to\infty}\int_\Gamma |f(\zeta)-R_k(\zeta)|^p\rho(\zeta)|d\zeta| = 0.$$

7. With the aid of the theorems given above one can investigate the question of the possibility of approximating the boundary values $f(\zeta)$ of functions analytic in the domain G and belonging to the classes E_δ, $\delta > 0$. (The definition of the classes E_δ in multiply connected domains, analogous to the definition of these classes for simply connected domains, is given in ([3]).) The results obtained in this way turn out to be similar to those obtained in considering approximation of the boundary values of the classes E_δ in simply connected domains by sequences of polynomials. (Formulations of the results for simply connected domains are given in our note ([2]).) We shall confine ourselves here to considering only the most important case, when the domain G belongs to the class S of domains satisfying V. I. Smirnov' s condition. The definition of the class S in the case of multiply connected domains, which is given in ([3,5]), reduces to the case of a simply connected domain: a domain $G \in S$ if each of the domains G_i, $i = 1, \dots, n$, belongs to the class S. (For the definition of the class S for simply connected domains, see, for example, ([1]).) It can be proved that this definition is equivalent to the following one: the harmonic function $\ln|\varphi'(w)|$, where $z = \varphi(w)$ maps the disk K onto G, is representable in K by Green' s formula. It is not difficult to see that if $f(z) \in E_\delta$ in the domain $G \in S$, then $f(z) \in D$. Then Theorems 4 and 5 immediately imply Theorem 9.

Theorem 9. *If $f(z) \in E_\delta$ in the domain $G \in S$, and the boundary values of this function $f(\zeta) \in L^p(d\sigma, \Gamma)$, $p > 0$, then there exists a sequence $\{R_k(\zeta)\}$ converging to $f(\zeta)$ in the metric $L^p(d\sigma, \Gamma)$.*

From Theorems 6 and 9 follows Theorem 10.

Theorem 10. *If $G \in S$, then for the existence of $\{\Pi_k(\zeta)\}$ satisfying the condition*

$$\lim_{k\to\infty}\int_\Gamma |f(\zeta)-\Pi_k(\zeta)|^p|d\zeta| = 0,$$

it is necessary and sufficient that $f(\zeta)$ coincide on Γ with the boundary values of a function $f(z)$, analytic in G, of the class E_p.

For $p > 1$ Theorem 10 was proved in ([5]).

8. In ([4,6]) the classes H_δ in multiply connected domains were studied. In any domain G, $H_\delta \subset D$ for any $\delta > 0$. Therefore Theorems 4 and 5 imply Theorem 11.

Theorem 11. *If $f(z) \in H_\delta$, $\delta > 0$, in the domain G and $f(\zeta) \in L^p(d\sigma, \Gamma)$, $p > 0$, then there exists $\{R_k(\zeta)\}$, converging on Γ in the metric $L^p(d\sigma, \Gamma)$ to $f(\zeta)$—the boundary values of $f(z)$.*

For $\delta = p$ and $\sigma(s) \equiv s$ this was proved by Rudin ([6]), moreover it was assumed that Γ is an analytic curve.

9. Results similar to those given above are obtained if instead of the space $L^p(d\sigma, \Gamma)$ one considers the space $C(\rho, \Gamma)$, consisting of functions $f(\zeta)$ continuous on Γ with norm $\|f\| = \max_\Gamma \{|f(\zeta)|\rho(\zeta)\}$, where $\rho(\zeta) \geqslant 0$ is a function continuous on Γ.

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Note: Figure translations are in progress. See original paper for figures.