MATHEMATICS

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Abstract

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EXISTENCE AND UNIQUENESS OF THE SOLUTION OF BOUNDARY-VALUE PROBLEMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 21 XI 1956)

In the works ([3], [9], [10]), mainly two-point boundary-value problems for nonlinear differential equations of the second order, solved with respect to the highest derivatives, were considered. Functional methods in related problems were applied in the works ([4], [5]).

1. Consider the equation

$$y^{(n)} = \varphi\left(x, y, y', \dots, y^{(n-1)}\right). \tag{1}$$

We shall say that φ satisfies condition L^0 if it is defined in $G: \ -\infty < x < +\infty, \ -\infty < y^{(k)} < +\infty$ $(k = 0, 1, 2, \dots, n-1)$, is continuous in all variables, continuously differentiable with respect to $y, y', \dots, y^{(n-1)}$, and $\varphi_{y^{(k)}} \geq 0$ $(k = 0, 1, 2, \dots, n-3)$, $\varphi_{y^{(n-2)}} > 0$ in G. If $S \geq \varphi_{y^{(k)}} \geq 0$ $(k = 0, 1, 2, \dots, n-3)$, $\varphi_{y^{(n-2)}} < \mu > 0$, where S and μ are some constant numbers, then we shall say that φ satisfies condition L_S^{μ}.

Lemma 1. Suppose we have a solution $y(x)$ of equation (1), whose function φ satisfies condition L^0, with initial data

$$y^{(k)}(a) = y_0^{(k)} \qquad (k = 0, 1, \dots, n-1), \tag{*}$$

and an n-times continuously differentiable function $z(x)$ on $[a, b]$ such that

$$z^{(n)}(x) \geq \varphi\left(x, z(x), z'(x), \dots, z^{(n-1)}(x)\right), \quad z^{(k)}(a) \geq y_0^{(k)} \quad (k = 0, 1, 2, \dots, n-2)$$

and

$$z^{(n-1)}(a) > y_0^{(n-1)}.$$

Then the differential inequality holds:

$$z^{(k)}(x) > y^{(k)}(x) \quad \text{on } (a, b] \qquad (k = 0, 1, 2, \dots, n-2).$$

The lemma is proved with the aid of the ideas of N. V. Azbelev ([1]) and B. N. Babkin ([2]) on Chaplygin differential inequalities. From this lemma it follows:

Theorem 1. If equation (1), where φ satisfies condition L^0, has a solution with boundary conditions

$$y^{(k)}(a) = y_0^{(k)} \qquad (k = 0, 1, 2, \dots, n-2)$$

and

$$y^{(i)}(b) = y_1^{(i)}, \quad \text{where } i \text{ is one of the numbers } 0, 1, 2, \dots, n-2,$$

then the solution is unique.

Theorem 2. Let φ satisfy condition L_S^μ and the following condition:

I. At every point of G the relation

$$|\varphi| \leq A\left(x, y, y', \dots, y^{(n-2)}\right)\left(y^{(n-1)}\right)^2 + B\left(x, y, y', \dots, y^{(n-2)}\right),$$

holds, where A, B are continuous and nonnegative in every closed domain G.

Then, for $b - a < \ln(1 + \mu/S)$, there exists a unique solution of equation (1) satisfying the boundary data $(*)$ and

$$y^{(n-2)}(b) = y_1^{(n-2)}.$$

Theorem 2 is proved with the aid of estimates of the derivatives of the solution and Lemma 1 by the method of analytic continuation of S. N. Bernstein' s solution with respect to a parameter. This theorem is a generalization of S. N. Bernstein' s theorem ([3]).

Theorem 3. Suppose that φ satisfies the conditions L_S^μ and condition I of Theorem 2, and that two functions $y_1(x)$, $y_2(x)$ are known, satisfying equation (1), the initial conditions $(*)$, and the inequalities

$$y_1^{(k)}(x) > y_2^{(k)}(x) \quad \text{on } (a, b] \qquad (k = 0, 1, 2, \dots, n-2).$$

Then there exists a unique solution satisfying the boundary conditions $(*)$ and

$$y^{(i)}(b) = y^{*(i)}, \qquad y_2^{(i)}(b) < y^{*(i)} < y_1^{(i)}(b),$$

where $b - a < \ln(1 + \mu/S)$.

The existence of $y_1(x)$ and $y_2(x)$ follows from Lemma 1 and Theorem 2.

2. We now consider the boundary-value problem for a system of n second-order differential equations

$$u'' = F(t, u, u'), \tag{2}$$

$$u(a) = \alpha, \qquad u(b) = \beta, \tag{3}$$

where u is a vector-function of the n-dimensional vector space E_n, defined on $[a, b]$, and α, β are new constant vectors.

Assume that $F(t, u, u')$ has the following properties:

a) it is defined in D:

$$a \le t \le b, \qquad \|u\|_{E_n},\ \|u'\|_{E_n} < +\infty;$$

b) for the matrix F_u, with arbitrary combinations of arguments, the condition

$$(F_u w, w) \ge m\|w\|^2_{E_n}, \qquad w \in E_n$$

is satisfied, and for the matrix $F_{u'}$, with arbitrary combinations of arguments, the relation

$$|(F_{u'} v, w)| \le m_1\|v\|^2_{E_n} + m_2\|w\|^2_{E_n}$$

holds, where $1 > m_1 > 0,\ m > m_2 > 0$, and these are constants;

c) for any points of D,

$$\|F(t, u, u')\|^2_{E_n} \le A_1(t, u)\|u'\|^2_{E_n} + B_1(t, u),$$

where A_1 and B_1 are continuous nonnegative functionals in any closed domain D.

First of all we prove one basic lemma.

Lemma 2. Given the linear equation

$$u'' = A(t)u + B(t)u'.$$

If for the matrices $A(t)$ and $B(t)$ the relations

$$(A(t)u, u) \ge m\|u\|^2_{E_n}, \qquad |(B(t)v, u)| \le m_1\|v\|^2_{E_n} + m_2\|u\|^2_{E_n}$$

hold for any elements u, v of E_n, where $m > m_2 > 0,\ 1 \ge m_2 > 0$, then the solution of this equation satisfying the zero boundary data is trivial.

Theorem 4. If $F(t, u, u')$ satisfies conditions a) and b) and there exists a solution of equation (2) with boundary conditions (3), then the solution is unique.

The proof follows from Lemma 2.

Theorem 5. If F satisfies properties a), b), and c), then equation (2) has a unique solution satisfying the boundary conditions (3).

For the proof of existence we introduce into equation (2) a numerical parameter λ in the following way:

$$u''_\lambda = \lambda F(t, u_\lambda, u'_\lambda), \qquad 0 \leqslant \lambda \leqslant 1, \qquad u_\lambda(a) = \alpha, \qquad u_\lambda(b) = \beta. \tag{4}$$

Passing to a system of integral equations, we obtain an integral operator equation equivalent to equation with parameter (4). By virtue of the compactness of the totality of solutions $\{u_\lambda(t)\}_{0\leqslant\lambda\leqslant 1}$, together with their first derivatives, and Lemma 2, to the integral operator equation there is applicable a theorem generalizing, in Banach space, the usual implicit-function theorem [7] as a whole. The solution of equation (4) is continued in λ from 0 to 1.

3. Consider an equation of the form

$$y'' = \varphi(x, y, y') + \alpha y. \tag{5}$$

Theorem 6. *Suppose that the following conditions are satisfied:*

1) φ is defined in the domain $\Omega:\ 0 \leqslant x \leqslant 1,\ |y|,\ |v| < +\infty$, and at every point of the domain $0 \leqslant x \leqslant 1,\ |y| < +\infty,\ |v| \leqslant L_0$ is continuously differentiable with respect to y and v, and $\varphi_y \geqslant 0$;

2) for arbitrary points in Ω,

$$|\varphi(x, y, v)| \leqslant \sigma_1 |y| + \sigma_2 |v| + \psi(x),$$

where $0 \leqslant \psi(x) \leqslant \sigma_3 m,\ |\varphi(x, 0, 0)| < m$;

3)

$$\sigma_1 \frac{1}{\alpha} + \sigma_2 \frac{1}{\sqrt{\alpha}} + \sigma_3 < 1, \qquad L_0 = \frac{1}{\sigma_2} e^{\sigma_3\{m(\frac{\sigma_1}{\alpha} + \sigma_3 + 1) + 1\}},$$

where $m, \alpha, \sigma_1, \sigma_2, \sigma_3$ are positive constants.

Then there exists a unique solution satisfying the conditions

$$y(0) - y(1) = 0, \qquad y'(0) - y'(1) = 0.$$

For the proof of the theorem we transform equation (5), taking as the unknown $y'' - \alpha y = u$, into the equation

$$u(x) = \varphi\left(x, \int_0^1 K(x,s)u(s)\,ds, \frac{d}{dx}\int_0^1 K(x,s)u(s)\,ds\right), \tag{6}$$

where $K(x,s)$ is the Green' s function of the equation $y' - \alpha y = u$ with the given boundary conditions. On the basis of the Leray-Schauder principle [(8)], the existence of a solution of equation (6) is proved inside the sphere of radius m in the space $C(0,1)$. The uniqueness of the solution follows from $\varphi_y \geqslant 1$.

4. We use the case to consider the behavior of the solution of a nonlinear parabolic equation of the type

$$u_t = u_{xx} - f(x,u,u_x), \tag{7}$$

$$u(0,t) = u(1,t) = u(x,0) = 0. \tag{8}$$

as t tends to infinity.

Take the ordinary differential equation

$$w'' = f(x,w,w'), \tag{9}$$

$$w(0) = w(1) = 0. \tag{10}$$

On the function f we impose the conditions:

a) $f(x,u,v)$ is defined in the domain $0 \leqslant x \leqslant 1$, $|u|$, $|v| < +\infty$, is continuously differentiable with respect to u, v, and $f_u \geqslant -c$, where c is a positive constant such that $0 < c < \pi^2$;

b) $f_v = k + \alpha(x)$, where k is a constant and $\alpha(x)$ is continuous in x;

c)

$$\frac{c}{\pi^2} + \frac{\max|\alpha(x)|}{2}\left\{1 + \frac{1}{\pi^2}\right\} < 1.$$

Theorem 7. If the function $f(x,u,v)$ satisfies conditions a), b), c), then the solution of equation (9) with boundary conditions (10) is unique.

Theorem 8. Suppose that equation (7), under conditions a), b), c), has a solution $u(x,t)$ satisfying the boundary conditions (8). Then, as t tends to infinity, $u(x,t)$ converges in the mean to the solution $w(x)$ of equation (9) with boundary conditions (10).

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REFERENCES

[1] N. V. Azbelev, *On the limits of applicability of S. A. Chaplygin' s theorem on differential inequalities*, Dissertation, 1954.
[2] B. N. Babkin, DAN, 58, No. 3 (1948).
[3] S. N. Bernstein, *Uspekhi Mat. Nauk*, 8, 32 (1941).
[4] G. D. Birkhoff, O. D. Kellogg, Trans. Am. Math. Soc., 23, 96 (1922).
[5] M. A. Krasnosel' skii, Izv. AN SSSR, Ser. Math., 20, 241 (1956).
[6] L. A. Lyusternik, DAN, 33, No. 1, 5 (1951).
[7] L. A. Lyusternik, V. N. Sobolev, *Elements of Functional Analysis*, 1951.
[8] H. Leray, J. Schauder, *Uspekhi Mat. Nauk*, No. 3–4 (1946).
[9] Z. F. Surikova, *On the boundary-value problem for ordinary differential equations*, Dissertation, Moscow, 1954.
[10] L. Tonelli, Ann. delle R. Sc. Norm. Sup. di Pisa, (2), 8, 75 (1939).

Note: Figure translations are in progress. See original paper for figures.