SovietRxiv

Soviet-era science, translated into English

BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

1957

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195701.99338

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

CHOU YU-LIN

BOUNDARY VALUE PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS

(Presented by Academician I. G. Petrovskii on 25 V 1957)

This note considers boundary value problems with nonlinear boundary conditions for a quasilinear parabolic equation with two independent variables, and the second boundary value problem for a nonlinear parabolic equation in the case of many independent variables. The solution of these problems is obtained in the large, i.e., for any prescribed interval of variation of the time t. The first boundary value problem for a quasilinear parabolic equation was considered in papers ([1–5]). To solve the boundary value problems considered in the present paper, we use Rothe' s method ([6]). To prove the convergence of the sequence of functions constructed by Rothe' s method to the solution of the boundary value problem, we use the method of auxiliary functions, analogously to how this was done in papers ([1,4]). In addition, in the present note a priori estimates are given for derivatives of solutions of quasilinear elliptic and parabolic equations, analogous to the estimates of S. N. Bernstein ([7]).

We shall use the following notation: by $C^{(n,\lambda)}(\overline{D})$ we shall denote the class of functions having, in the closed domain $\overline{D}$, derivatives of order n satisfying a Hölder condition with exponent λ. We shall say that the closed domain $\overline{D}$ belongs to the class $A^{(n,\lambda)}$ if the boundary $\mathfrak{F}D$ of the domain D admits a local parametric representation by means of functions belonging to the class $C^{(n,\lambda)}$.

Theorem 1. *In the rectangle $R\{0 \le t \le T;\ 0 \le x \le X\}$ there exists a unique solution of the equation*

$$\frac{\partial^2 u}{\partial x^2} = A(x,t,u)\frac{\partial u}{\partial t} + F\left(x,t,u,\frac{\partial u}{\partial x}\right), \tag{1}$$

continuous together with the first derivative with respect to x in R, possessing continuous derivatives of second order with respect to x and first order with respect to t inside R, and satisfying the boundary conditions

$$u(x,0) = \varphi(x), \qquad u'_x(0,t) = \psi_0(t,u(0,t)), \qquad u'_x(X,t) = \psi_1(t,u(X,t)),$$

provided the following assumptions are fulfilled:

SovietRxiv

1) *In the rectangle R, for all values of u,*

$$A(x,t,u) \geq a > 0, \qquad F'_u(x,t,u,0) \geq c,$$

where a and c are certain constants.

2) *In the domain*

$$G\{0 \leq x \leq X;\ 0 \leq t \leq T;\ |u| \leq M_1;\ |p| < \infty\},$$

where

$$M_1 = e^{aT}\left\{\max|\varphi(x)| + \frac{2T}{a}\max|F(x,t,0,0)|\right\}$$

and a is such that $\frac{1}{2}aa+c > 0$, there exist continuous derivatives of second order with respect to x, u, p of the functions $A(x,t,u)$ and $F(x,t,u,p)$, satisfying a Lipschitz condition with respect to t, u, p.

3) The function $F(x,t,u,p)$ and its derivatives with respect to x, u in the domain G have order of growth in p less than two, while the function $F'_p(x,t,u,p)$ has order of growth less than one.

4) The function $\varphi(x)$ on the interval $[0, X]$ has a continuous derivative of third order. The functions $\psi_0(t,u)$ and $\psi_1(t,u)$ satisfy the conditions $\psi'_{0u}(t,u) \geqslant 0$, $\psi'_{1u}(t,u) \leqslant 0$, $\psi_0(t,0) \equiv \psi_1(t,0) \equiv 0$. In the domain $\{0 \leqslant t \leqslant T;\ |u| \leqslant M_1\}$ the functions $\psi_0(t,u)$, $\psi_1(t,u)$ have continuous derivatives with respect to u of second order and with respect to t of first order.

5) The compatibility conditions are fulfilled

$$\varphi'(0) = \psi_0(0,\varphi(0)), \qquad \varphi'(X) = \psi_1(0,\varphi(X)). \tag{3}$$

Theorem 2. In the rectangle R there exists a unique solution of equation (1), continuous in R together with the derivatives entering the equation, and satisfying the boundary conditions

$$u(x,0) = \varphi(x), \qquad u'_t(0,t) = \psi_0(t, u(0,t), u'_x(0,t)),$$
$$u'_t(X,t) = \psi_1(t, u(X,t), u'_x(X,t)), \tag{6}$$

if the following assumptions are fulfilled:

1) In the rectangle R, for all values of u, p, $A(x,t,u) \geqslant a > 0$, $F'_u(x,t,u,p) \geqslant c$,

$$\psi'_{0p}(t,u,p) \geqslant \alpha_0 > 0, \qquad \psi'_{0u}(t,u,0) \leqslant \beta_0, \qquad \psi_0(t,0,0) \equiv 0,$$

SovietRxiv

$$\psi'_{1p}(t,u,p) \leqslant -\alpha_1 < 0, \qquad \psi'_{1u}(t,u,0) \leqslant \beta_1, \qquad \psi_1(t,0,0) \equiv 0, \qquad (7)$$

where $a, c, \alpha_0, \alpha_1, \beta_0, \beta_1$ are certain constants.

2) In the domain $G\{0 \leqslant x \leqslant X;\ 0 \leqslant t \leqslant T;\ |u| \leqslant M_1;\ |p| < \infty\}$, where

$$M_1 = \max\left\{ e^{aT} \max \frac{|F(x,t,0,0)|}{{}^1\!/_2 aa + c};\ e^{aT} \max |\varphi(x)| \right\}$$

and a is such that ${}^1\!/_2 aa + c > 0$, there exist continuous derivatives of third order with respect to x, u, p of the functions $A(x,t,u)$, $F(x,t,u,p)$; moreover the functions $A(x,t,u)$, $F(x,t,u,p)$, together with their derivatives up to second order with respect to x, u, p, satisfy a Lipschitz condition with respect to t.

3) The functions F, F'_x, F'_u in the domain G have order of growth in p less than two, while the function F'_p has order of growth less than one.

4) The function $\varphi(x)$ on the interval $[0, X]$ has a continuous derivative of fourth order, and the functions $\psi_0(t,u,p)$, $\psi_1(t,u,p)$ have, with respect to u, p, continuous derivatives of second order and, with respect to t, of first order; moreover the functions $\psi_0(t,u,p)$, $\psi_1(t,u,p)$ have order of growth in p less than two.

5) The compatibility conditions are fulfilled

$$\varphi''(0) = A(0,0,\varphi(0))\psi_0(0,\varphi(0),\varphi'(0)) + F(0,0,\varphi(0),\varphi'(0)),$$

$$\varphi''(X) = A(X,0,\varphi(X))\psi_1(0,\varphi(X),\varphi'(X)) + F(X,0,\varphi(X),\varphi'(X)).$$

Theorem 3. In the cylinder $Q\{0 \leqslant t \leqslant T;\ x \in \overline{D}\}$ there exists a unique solution $u(x,t)$ of the equation

$$\sum_{i,j=1}^{N} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,t,u) \frac{\partial u}{\partial x_i} + f(x,t,u) = \frac{\partial u}{\partial t}, \qquad (2)$$

continuous in Q together with its first derivatives with respect to x_k,

possessing continuous derivatives of first order with respect to t and of second order with respect to x_k inside the cylinder Q and satisfying the boundary conditions

$$u(x,0) = u_0(x) \qquad \text{for } x \in \overline{D},$$

$$\sum_{i,j=1}^{N} a_{ij}(x,t) \cos(\gamma, x_i) \frac{\partial u}{\partial x_j} + \psi(x,t)u = 0 \quad \text{for } x \in \mathfrak{F}D, \quad 0 \leq t \leq T, \qquad (3)$$

where γ is the inner normal to the boundary $\mathfrak{F}D$, if the following assumptions are fulfilled:

SovietRxiv

1) In the cylinder Q the inequality

$$\sum_{i,j=1}^{N} a_{ij}\eta_i\eta_j \geq a\sum_{i=1}^{N}\eta_i^2 \quad (a>0)$$

holds for $(x,t) \in Q$ and all values, and $f'_u(x,t,u) \leq c$, where a, c are constants.

2) The functions $a_{ij}(x,t)$, as functions of x_k, belong to the class $C^{(5,\lambda)}(\overline{D})$, and the functions $b_i(x,t,u)$, $f'_u(x,t,u)$, $f(x,t,0)$ in the region

$$H\{0 \leq t \leq T;\ x \in D;\ |u| \leq M_1\},$$

where

$$M_1 = e^{aT}\left\{\max_{\overline{D}}|u_0(x)| + 2T\max_{Q}|f(x,t,0)|\right\},$$

$\frac{1}{2}a - c > 0$, $\alpha > 0$, belong to the class $C^{(4,\lambda)}$ as functions of x_k, u. In addition, the functions $a_{ij}(x,t)$, together with their derivatives up to third order with respect to x_k, have derivatives with respect to t satisfying a Lipschitz condition in t in Q, and the functions $b_i(x,t,u)$, $f'_u(x,t,u)$, $f(x,t,0)$, together with the second derivatives with respect to x_k, u, satisfy a Lipschitz condition in t in H.

3) The region $\overline{D}$ belongs to the class $A^{(6,\lambda)}$.

4) The function $u_0(x)$ belongs to the class $C^{(6,\lambda)}(\overline{D})$. The function $\psi(x,t) \geq 0$ belongs to the class $C^{(5,\lambda)}\mathfrak{F}(D)$ and, together with the third derivatives with respect to the local coordinates of the boundary of the region, has a derivative with respect to t satisfying a Lipschitz condition in t.

5) The compatibility condition is satisfied:

$$\sum_{i,j=1}^{N} a_{ij}(x,0)\cos(\gamma,x_i)\frac{\partial u_0(x)}{\partial x_j} + \psi(x,0)u_0(x) = 0 \quad \text{for } x \in \mathfrak{F}D.$$

It is proved that the solution of problem (2), (3) can be obtained as the limit, as $\Delta t \to 0$, of the solutions of the following system of elliptic equations:

$$\sum_{i,j=1}^{N} a_{ij}(x,n\Delta t)\frac{\partial^2 u(x,n\Delta t)}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,n\Delta t,u(x,(n-1)\Delta t))\frac{\partial u(x,n\Delta t)}{\partial x_i}$$

$$+\,c(x,n\Delta t,u(x,(n-1)\Delta t))u(x,n\Delta t)+f(x,n\Delta t) = \frac{u(x,n\Delta t)-u(x,(n-1)\Delta t)}{\Delta t};$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \overline{D},$$

SovietRxiv

$$\sum_{i,j=1}^{N} a_{ij}(x, n\Delta t) \cos(\gamma, x_i) \frac{\partial u(x, n\Delta t)}{\partial x_j} + \psi(x, n\Delta t) u(x, n\Delta t) = 0 \quad \text{for } x \in \mathfrak{F}D,$$

where

$$c(x, t, u) = \int_0^1 f'_u(x, t, \tau u)\, d\tau.$$

The proof of the existence of a solution of this system belonging to the class $C^{(6,\lambda)}(\overline{D})$ is based on the following theorem, analogous to Schauder' s theorem ([8]).

Theorem 4. Let the domain $\overline{D}$ belong to the class $A^{(n+1,\lambda)}$, $n \geqslant 3$, $0 < \lambda < 1$. Then the second boundary-value problem for the equation

$$\sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial u}{\partial x_i} + c(x) u = f(x)$$

with boundary condition

$$\sum_{i,j=1}^{N} a_{ij}(x) \cos(\gamma, x_i) \frac{\partial u}{\partial x_j} + \psi(x) u = \varphi(x)$$

in the domain D has a unique solution $u(x)$ belonging to the class $C^{(n,\lambda)}(\overline{D})$, if the coefficients $a_{ij}(x) \in C^{(n,\lambda)}(\overline{D})$, $b_i(x)$, $c(x)$, $f(x) \in C^{(n-2,\lambda)}(\overline{D})$, $\psi(x)$, $\varphi(x) \in C^{(n-1,\lambda)}(\mathfrak{S}D)$, the functions $\psi(x)$, $c(x)$ are nonnegative, and at least one of them is not identically equal to zero.

Theorem 5. Let $u(x,t) \in C^{(n+2)}(Q)$ and be a bounded solution in the cylinder Q of the quasilinear parabolic equation

$$L(u) \equiv \sum_{i,j=1}^{N} a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + f\left(x, t, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right) \equiv \frac{\partial u}{\partial t}$$

$(a_{ij}, f \in C^{(n)})$, and let the functions

$$|f(x, t, u, p_1, \dots, p_N)|, \quad |f'_{x_k}(x, t, u, p_1, \dots, p_N)|,$$

$$|f'_u(x, t, u, p_1, \dots, p_N)|$$

not exceed $p^2 \varepsilon(p)$, and

$$|f'_{p_i}(x, t, u, p_1, \dots, p_N)| \leqslant p \varepsilon(p) \quad (i, k = 1, 2, \dots, N),$$

SovietRxiv

where

$$p = \left[\sum_{k=1}^{N} p_k^2 \right]^{1/2},$$

and $\varepsilon(p)$ is some monotonically decreasing function tending to zero as $p \to \infty$. Then the inequality

$$\left| \frac{\partial^n u(x,t)}{\partial x_{i_1} \cdots \partial x_{i_n}} \right| \leqslant C$$

holds in any closed domain strictly interior with respect to Q,

$$\{x \in \overline{D}_1 \subset D,\ 0 < \delta \leq t \leq T\},$$

where C depends on $\max_Q |u(x,t)|$, n, δ, and the distance between the boundaries of the domains D_1 and D.

In the case of two independent variables this theorem is valid for an equation of the form

$$\frac{\partial^2 u}{\partial x^2} = A(x,t,u)\frac{\partial u}{\partial t} + F\left(x,t,u,\frac{\partial u}{\partial x}\right).$$

An analogous theorem has also been proved for a quasilinear elliptic equation of the form $L(u) = 0$.

In conclusion I express my deep gratitude to my scientific adviser O. A. Oleinik.

Moscow State University
named after M. V. Lomonosov

Received
25 V 1957

CITED LITERATURE

1. O. A. Oleinik, T. D. Venttsel, *Matem. sborn.*, **41**, No. 1, 105 (1957).
2. C. Ciliberto, *Ricerche mat.*, **3**, 129 (1954).
3. O. A. Ladyzhenskaya, *DAN*, **107**, No. 5, 636 (1956).
4. T. D. Venttsel, *Matem. sborn.*, **41** (83), No. 4 (1957).
5. E. Gagliardo, *Ricerche mat.*, **5**, 239 (1956).
6. E. Rothe, *Math. Ann.*, **102**, 650 (1930).
7. S. N. Bernstein, *DAN*, **18**, No. 7, 385 (1938).

8. J. Schauder, *Math. Zs.*, **38**, 257 (1934).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.