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Abstract

Full Text

MATHEMATICS

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ON THE BEHAVIOR OF SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS IN A NEIGHBORHOOD OF AN ISOLATED STATIC SOLUTION

(Presented by Academician N. N. Bogolyubov, 25 II 1957)

Consider the system of differential equations

$$\frac{dx}{dt} = X(x) + \varepsilon X^*(t, x, \varepsilon), \tag{1}$$

where ε is a small positive parameter.

We shall make the following assumptions:

a) The system of unperturbed equations

$$\frac{dx}{dt} = X(x) \tag{2}$$

has an isolated static solution corresponding to the equilibrium point

$$x = 0, \quad X(0) = 0 \quad (X'_x(0) \neq 0). \tag{3}$$

b) The functions $X(x) + \varepsilon X^*(t, x, \varepsilon)$ in the domain

$$-\infty < t < \infty, \quad x \in U_{\sigma_0}, \quad 0 < \varepsilon < \varepsilon_0, \tag{4}$$

where U_{σ_0} is the σ_0-neighborhood of the point $x = 0$, are periodic in t with period 2π and have bounded and uniformly continuous derivatives with respect to x, ε of arbitrary order.

c) For the variational equations

$$\frac{d\delta x}{dt} = X'_x(0)\delta x, \tag{5}$$

corresponding to the static solution (3), the characteristic equation

$$|J_n z - A| = 0 \quad (A = X'_x(0)) \tag{6}$$

has a pair of purely imaginary roots ($z_1 = i\omega,\ z_2 = -i\omega$), while the remaining roots ($z_3, \dots, z_n$) have negative real parts.

Under these assumptions it can be proved that system (1) has a unique local integral manifold, whose parametric representation depends on two arbitrary constants, and to which, as time proceeds, all solutions will tend whose initial values are sufficiently close to the indicated integral manifold.

Let us write system (1) in the form

$$\frac{dx}{dt} = X'_x(0)x + \{\overline{X}(x) + \varepsilon X^*(t, x, \varepsilon)\}, \tag{7}$$

where $\overline{X}(x)$ begins with terms of second order with respect to x, and, consequently,

respectively, there will also be found $\overline{C}$ and δ_0 such that, for $|x| < \delta_0$, the inequality

$$|\overline{X}(x)| \leq \overline{C}x^2 \qquad (\delta_0 < \sigma_0). \tag{8}$$

will hold.

As is known, the general solution of the system of equations (5) has the form

$$\delta x = C_1 A e^{i\omega t} + C_2 B e^{-i\omega t} + D\overline{h},$$

where C_1, C_2 are arbitrary constants (C_1, C_2 are complex conjugates); A, B are constant vectors; D is a constant matrix; $\overline{h}$ is a linear combination of products of polynomials by decaying exponentials, containing $n-2$ arbitrary constants.

Making in the system (7) the change of variables

$$x = A\xi + B\xi^* + Dh, \tag{9}$$

we obtain

$$\begin{aligned} \frac{d\xi}{dt} &= i\omega\xi + P(t, \xi, \xi^*, h, \varepsilon), \\ \frac{d\xi^*}{dt} &= -i\omega\xi^* + Q(t, \xi, \xi^*, h, \varepsilon), \\ \frac{dh}{dt} &= Hh + R(t, \xi, \xi^*, h, \varepsilon), \end{aligned} \tag{10}$$

where H is a square matrix of order $n-2$, for which the equation

$$|J_{n-2}z - H| = 0$$

has all roots with negative real parts.

At the same time one can specify such a ρ_1

$$\left(\rho_1 < \frac{\delta_0}{|A|+|B|+|D|}\right),$$

that for any $|\xi| < \rho_1$, $|\xi^*| < \rho_1$, $|h| < \rho_1$ we shall have $|x| < \delta_0$, and the functions $P(t,\xi,\xi^*,h,\varepsilon)$, $Q(t,\xi,\xi^*,h,\varepsilon)$, $R(t,\xi,\xi^*,h,\varepsilon)$, for $h=0$ in the domain

$$-\infty < t < \infty, \qquad |\xi| < \rho, \qquad |\xi^*| < \rho, \qquad |h| < \rho, \qquad 0 < \varepsilon < \varepsilon_0, \tag{11}$$

tend to zero as $\varepsilon \to 0$, $\rho^2 \to 0$ $(\rho < \rho_1)$.

After this, for the basic equation (1) we can formulate the following theorem.

Theorem. *Suppose that, for the system of equations (1), conditions a), b), c) are satisfied. Then one can specify positive numbers* $\varepsilon', \delta_1, \rho_1$

$$\left(\delta_1 < \rho_1 < \frac{\delta_0}{|A|+|B|+|D|}\right),$$

such that, for any positive $\varepsilon < \varepsilon'$, $\delta < \delta_1, \rho < \rho_1$, *the following assertions will be valid:*

1. *Equation (1) has a unique two-dimensional local* integral manifold* $\mathfrak{M}_t$, *lying in the domain* $U_{\sigma_1}^{\varepsilon'}$, *where* $\sigma_1 = \rho_1\{|A|+|B|+|D|\}$.
2. *On the local integral manifold, equation (1) is equivalent to the system*

$$\begin{aligned} \frac{d\xi_t}{dt} &= i\omega\xi_t + P_1(t,\xi_t,\xi_t^*,\varepsilon), \\ \frac{d\xi_t^*}{dt} &= -i\omega\xi_t^* + Q_1(t,\xi_t,\xi_t^*,\varepsilon), \end{aligned} \tag{12}$$

* We shall call $\mathfrak{M}_t$ a local integral manifold if, from the relation $x(t_0,\xi_0,\xi_0^*,\varepsilon) \in \mathfrak{M}_t$, $|\xi_0| < \rho_1$, $|\xi_0^*| < \rho_1$, valid at the moment $t = t_0$, it follows that $x(t,\xi_t,\xi_t^*,\varepsilon) \in \mathfrak{M}_t$ for any t as long as $|\xi_t| < \rho_1$, $|\xi_t^*| < \rho_1$.

where the functions $P_1(t,\xi_t,\xi_t^*,\varepsilon)$, $Q_1(t,\xi_t,\xi_t^*,\varepsilon)$ are defined in the domain (11), have bounded and uniformly continuous derivatives, and are periodic in t with period 2π.

3. The local integral manifold $\mathfrak{M}_t$ admits a parametric representation of the form

$$x^*(t) = A\xi_t + B\xi_t^* + Dh_t^M = F(t, \xi_t, \xi_t^*, \varepsilon), \tag{13}$$

where $F(t, \xi_t, \xi_t^*, \varepsilon)$ is defined in the domain

$$-\infty < t < \infty, \qquad |\xi_t| < \rho_1, \qquad |\xi_t^*| < \rho_1, \qquad 0 < \varepsilon < \varepsilon', \tag{14}$$

has bounded and uniformly continuous derivatives with respect to $\xi_t, \xi_t^*, \varepsilon$ of any order, and has period 2π with respect to t.

4. The local integral manifold attracts to itself (as long as $|\xi_t| < \rho_1$, $|\xi_t^*| < \rho_1$) any solutions of system (1) $x(t)$ whose initial values belong to U_{δ_1}, i.e., the inequalities

$$|x(t) - x^*(t)| \le C_1(\varepsilon, \delta^2) e^{-\gamma(t-t_0)}; \tag{15}$$

$$\left| \frac{d\xi_t}{dt} - i\omega\xi_t - \varepsilon P_1(t, \xi_t, \xi_t^*, \varepsilon) \right| \le C_2(\varepsilon, \delta^2) e^{-\gamma(t-t_0)};$$

$$\left| \frac{d\xi_t^*}{dt} + i\omega\xi_t^* - \varepsilon Q_1(t, \xi_t, \xi_t^*, \varepsilon) \right| \le C_3(\varepsilon, \delta^2) e^{-\gamma(t-t_0)}. \tag{16}$$

Let us consider the special case of system (1) when $X(x) = Px$, where $P = \|p_{ik}\|$ is a constant matrix of order n, i.e., we shall consider the system

$$\frac{dx}{dt} = Px + \varepsilon X^*(t, x, \varepsilon), \tag{17}$$

where ε is a small positive parameter; X^* is an n-dimensional vector of Euclidean space. Here $X^*(t, x, \varepsilon)$ is a periodic function of t with period 2π, having bounded and uniformly continuous derivatives of any order in the domain

$$-\infty < t < \infty, \qquad 0 < \varepsilon < \varepsilon_0, \qquad |x| < N, \qquad N = \text{const}. \tag{18}$$

Suppose that the characteristic equation for the system of first approximation $dx/dt = Ax$ has a pair of purely imaginary roots $\pm i\omega$, while the remaining roots $z_3, \dots, z_n$ have negative real parts.

If these conditions are fulfilled, there is a theorem which, analogously to the preceding theorem, establishes the existence and uniqueness of an integral manifold for system (17), to which, as time passes, any solutions of system (17) will tend. However, in contrast to the preceding theorem, which is local in character, in the present case this attraction property already holds for arbitrary (finite) values of $x(t)$.

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REFERENCES

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2. O. B. Lykova, Reports of the Academy of Sciences of the Ukrainian SSR, No. 2 (1957).

Note: Figure translations are in progress. See original paper for figures.