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Abstract

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INTEGRAL REPRESENTATIONS OF OPERATOR-ANALYTIC FUNCTIONS OF ONE INDEPENDENT VARIABLE

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Let

$$L = \frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_0(x) \quad (1)$$

be an ordinary linear differential operator with continuous complex (in particular, real) coefficients, given on an interval (a, b) $(-\infty \leq a < b \leq +\infty)$ of the real number line;

$$M = \frac{d^n}{dw^n} + q_{n-1}(w)\frac{d^{n-1}}{dw^{n-1}} + \cdots + q_0(w) \quad (2)$$

an operator with coefficients analytic in a domain G of the complex w-plane.

In the note ([1]) a topological ring A_{L,x_0} of functions was constructed, each of which is operator-analytic with respect to z in a (its own) neighborhood of a certain (arbitrary but fixed) point $x_0 \in (a, b)$, and it was also indicated that the corresponding ring A_{M,w_0} for the operator M is, as a set, simply the set A_w of functions analytic in a neighborhood of the point w_0. On this basis the equivalence of all ordinary operators of equal order (both of the form (1) and of the form (2)) was proved: a transformation of one operator L into another was constructed, i.e., such a transformation T which: 1) maps one ring A_{L,x_0} isomorphically onto another; 2) carries L into the corresponding other operator.

In the present note an integral form is constructed for the transformation $T = T_{M,w_0;L,x_0}$ (in this sense) of the operator M into the operator L—according to the scheme originating in the works of J. Delsarte ([2]) and A. Ya. Povzner ([3]), devoted to second-order operators. Thus every function $g(x) \in A_{L,x_0}$, i.e. L-analytic in a neighborhood of x_0, receives an integral representation $g(x) = Tf(w)$ through a function $f(w)$, analytic in a neighborhood of w_0. This scheme is based on the theory of the Cauchy problem for the partial differential equation

$$MF(w,x) = LF(w,x), \tag{3}$$

which must be solved with the initial values

$$F(w,x_0) = f_0(w), \dots, \left.\frac{\partial^{n-1}F(w,x)}{\partial x^{n-1}}\right|_{x=x_0} = f_{n-1}(w), \tag{4}$$

analytic in the domain G.

§ 1. **Local integral representation.** Using the results of the note ([1]), one can obtain the following theorem:

Theorem 1. For each $w_0 \in G$, the solution $F(w,x)$ of problem (3), (4) exists in the complex-real cylinder $C(|w-w_0| < \alpha,\ |x-x_0| < \beta)$ and can be represented in the form of the sum of the double series

$$F(w,x) = \sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty} a_{\mu,m} g_\mu(w,w_0) f_m(x,x_0), \tag{5}$$

where $\{g_\mu(w,w_0)\}_0^\infty$ is an M-basis at the point w_0; $\{f_m(x,x_0)\}_0^\infty$ is an L-basis at the point x_0, and the coefficients satisfy the inequalities

$$|a_{\mu,m}| \le C_1^\mu C_2^m \mu! m! \tag{6}$$

for all $\mu, m = 0, 1, 2, \dots$.*
The series (5) converges absolutely and uniformly inside C and permits termwise application of the operators $(\partial^\rho/\partial w^\rho)M^\chi$, $(\partial^r/\partial x^r)L^q$ $(\rho, r = 0, 1, \dots, n-1;\ \chi, q = 0, 1, 2, \dots)$ in any order. Hence, from (6) one may obtain the following result:

Theorem 2. The function $F(w,x)$ of the preceding theorem and its $(\partial^\rho/\partial w^\rho)M^\chi$-images $(\rho \le n-1)$, for fixed w, are L-analytic with respect to x (in the neighborhood $|x-x_0| < \beta$).

Fix $w_0 \in G$, consider some function $f(w) \in A_w$, and find the solution $F(w,x) = F_k(w,x)$ of problem (3), (4) for $f_k(w) = f(w)$, $f_s(w) = 0$ $(s = 0, 1, \dots, n-1;\ s \ne k)$. The functions

$$c_{\mu,k}(x) \equiv c_{\chi n+\rho,k}(x) = \left.\frac{\partial^\rho}{\partial w^\rho} M^\chi F_k(w,x)\right|_{w=w_0}$$

$(\mu = \chi n + \rho)$ will then be L-analytic in a neighborhood of x_0, i.e., will belong to the ring A_{L,x_0}. This defines the operators $T_{\mu,k} f(w) = c_{\mu,k}(x)$ $(\mu = 0, 1, 2, \dots;\ k = 0, 1, \dots, n-1)$, transforming A_{M,w_0} into A_{L,x_0} and constituting a matrix $\widetilde{T}$ with infinitely many rows and n columns, the elements of which, under the action of the operator M on the right or L on the left, are shifted downward by n rows; that is, in general:

$$T_{\rho,k}M^{\chi} = L^{\chi}T_{\rho,k} = T_{\chi n+\rho,k}{}^{**}. \tag{7}$$

By virtue of (7), the matrix $\widetilde{T}$ naturally decomposes into square cells $\widetilde{T}_0, \widetilde{T}_1, \ldots$ (from top to bottom) of $n \times n$ operators in each cell.

Comparing the properties of the operators $T_{\rho,k}$ $(\rho, k = 0, 1, \ldots, n-1)$ of the upper cell $\widetilde{T}_0$ with the properties of the operator $T = T_{M,w_0;\,L,x_0}$, constructed in (1), we arrive at the following basic result:

Theorem 3. The trace $T_{0,0} + \cdots + T_{n-1,n-1}$ of the cell $\widetilde{T}_0$ coincides with the transformation T, which isomorphically maps the ring A_{M,w_0} onto the ring A_{L,x_0} and carries the operator M into the operator L.

Thus, a local "integral" representation $g(x) = Tf(w)$ has been obtained for L-analytic functions $g(x)$, which is integral only in the sense that it is composed of solutions ("integrals") of the Cauchy problem (3), (4).

§ 2. The domain of dependence of the integral representation. Applying the method of increasing the number of independent variables ([5]) to the solution of problem (3), (4), in particular, in constructing each function

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* With the possible exception of $\mu = m = 0$; here C_1 and C_2 are some constants; one may take $C_1 = C_2$.

** In these relations, apparently, the foundations are laid for extending to the operators (1) the theory of the generalized shift operators of Delsarte ([2])–Levitan ([4]).

$F_k(w, x)$, we obtain an L-analytic continuation of these functions (in x) and thereby an L-analytic continuation of the representation $g(x) = Tf(w)$:

Theorem 4. If $f(w)$ is regular in the disk $|w - w_0| < R$, then the function $g(x) = Tf(w)$ admits an L-analytic continuation* to the interval $(x_0 - R,\ x_0 + R) \cap (a, b)$.

In the case of sufficiently smooth coefficients of the operator L, applying to each function $F_k(w, x)$ the generalized Riemann formula (see ([5]), § 3), we obtain the following integral (in the ordinary sense) form of the representation $g(x) = Tf(w)$:

Theorem 5. If each coefficient $p_k(x)$ of the operator L is continuously differentiable k times $(k = 0, 1, \ldots, n-1)$, then

$$g(x) = Tf(w) = \sum_I \int_{\Omega_I} \cdots \int K_I(w_0, x; t_I) f(w_{0,I})\, dt_{I_\alpha}. \qquad (8)$$

Here, for brevity, by I is denoted an arbitrary nonempty combination (subset) of numbers $i_0 < i_1 < \dots < i_m$ from the set $1, 2, \dots, n$; $\sum_I$ extends over all these combinations; $K_I(w, x; t)$ is a linear combination of derivatives

$$\frac{\partial^k}{\partial w^k} \frac{\partial^s}{\partial t_{\sigma_1} \cdots \partial t_{\sigma_s}} v(w, x; t)$$

of the Riemann function $v(w, x; t) = v(w, x; t_1, \dots, t_n)$, where the set of indices $\sigma_1 < \sigma_2 < \dots < \sigma_s$ runs through I and $k + s \le m$; the introduction of t_I instead of t as an argument means that the variables t_i with numbers $i \notin I$ are set equal to zero; the coefficients of these linear combinations $K_I(\cdots)$ are regular functions of w; $w_{0,I} = w_0 \pm \sum_{i \in I} t_i \varepsilon_i$**, where $\varepsilon_1 = 1, \dots, \varepsilon_n$ are all roots of the n-th degree of 1; finally, the domain of integration Ω_I is the m-dimensional face ($t_i = 0$ for $i \notin I$), corresponding to I, of the simplex ($t_1 \ge 0, \dots, t_n \ge 0,\ t_1 + \dots + t_n = |x - x_0|$) in the n-dimensional space of characteristic variables $t_1, \dots, t_n$; in integration one of the variables, for example t_{i_α}, is excluded, and then dt_{I_α} is an abbreviated notation for the product of all differentials $dt_{i_0}, \dots, dt_{i_m}$, except dt_{i_α}.

The nonintegral term of formula (8), corresponding to $m = 0$***, is equal to

$$\exp\left\{ \frac{1}{n} \int_{w_0}^{w_0 + x - x_0} q_{n-1}(w)\, dw + \frac{1}{n} \int_x^{x_0} p_{n-1}(t)\, dt \right\} f(w_0 + x - x_0). \qquad (9)$$

For the simplest operator $M = d^n/dw^n$, the coefficients of the linear combinations $K_I(\cdots)$ can be computed explicitly in the form of functions of the roots $\varepsilon_1, \dots, \varepsilon_n$ and of the values $p_k^{(\nu)}(x_0)$ ($\nu = 0, 1, \dots, k;\ k = 1, 2, \dots, n-1$). The resulting general formula, which we do not write out for lack of space, for $n = 2,\ w_0 = 0,\ x_0 = 0,\ L = d^2/dx^2 + q(x)$, after a change of variable of integration, coincides with formula (4) in the note by A. Sh. Blokh ([6]). For $n = 3,\ w_0 = 0,\ x_0 = 0,\ L = d^3/dx^3 + p(x) d/dx + q(x)$, we obtain

* Unique by Theorem 6 ([1]).

** Plus for $x \ge x_0$, minus for $x < x_0$.

*** It would seem that there should be n such terms, corresponding to the values $i_0 = 1, 2, \dots, n$; but they are

$$g(x) = f(x) + \int_0^{|x|} K_2(x;t)\, f\big(\pm[\varepsilon_1 t + \varepsilon_2(|x|-t)]\big)\, dt + \int_0^{|x|} K_3(x;t)\, f\big(\pm[\varepsilon_1 t + \varepsilon_3(|x|-t)]\big)\, dt + \int_0^{|x|} ds \int_0^{|x|-s} K_{23}(x;s,t)\, f\big(\pm[\varepsilon_1 s + \varepsilon_2 t + \varepsilon_3(|x|-s-t)]\big)\, dt. \quad (10)$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\varepsilon_3 = \bar{\varepsilon}_2$; $K_2(x;t) = \partial v(x;t_1,t_2,0)/\partial t_2$ for $t_1 = t$, $t_2 = |x| - t$; $K_3(x;t) = \partial v(x;t_1,0,t_3)/\partial t_3$ for $t_1 = t$, $t_3 = |x| - t$; $K_{23}(x;s,t) = \partial^2 v(x;t_1,t_2,t_3)/\partial t_2 \partial t_3 + p(0)v$ for $t_1 = s$, $t_2 = t$, $t_3 = |x| - s - t$. Here the Riemann function $v = v(x;t_1,t_2,t_3)$ does not depend on w, and, by virtue of its symmetry in t_1, t_2, t_3, the functions $K_2(x;t)$ and $K_3(x;t)$ are equal to each other. Thus, in the case $n = 3$ one may say that the integrals in the integral representation of L-analytic functions are taken over the triangle with vertices $x\varepsilon_1 = x$, $x\varepsilon_2$, $x\varepsilon_3$ in the complex w-plane, along two of its sides with common vertex x, and "over the vertex" x.

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Note: Figure translations are in progress. See original paper for figures.