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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MATHEMATICS

Yu. Smirnov

AN EXAMPLE OF A ONE-DIMENSIONAL* NORMAL SPACE NOT CONTAINED IN ANY ONE-DIMENSIONAL BICOMPACTUM

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V. Hurewicz proved that every space with a countable base has a bicompact extension (with a countable base) of the same dimension (1). This result was generalized by W. H. Dowker to arbitrary normal spaces for the dimension dim, defined by means of coverings (2). N. B. Vedenisov obtained an analogous result for the large inductive dimension Ind (3). For the small inductive dimension ind an analogous proposition was obtained by N. B. Vedenisov only for zero-dimensional spaces (4). The example given here shows that already for one-dimensional normal spaces there need not exist one-dimensional bicompact extensions.

§ 1. The auxiliary space S^{**}.

A. Let c be the cardinality of the continuum, and let $\omega(c)$ be the least ordinal number of cardinality c. Let Γ be the ordered set of all possible pairs αx, where α is any ordinal number less than $\omega(c)$, and x is some number of the half-interval $[0;1)$. The order in Γ is given as follows: we take $\alpha x < \alpha' x'$ if $\alpha < \alpha'$, or if $\alpha = \alpha'$ but $x < x'$. Each pair $\alpha 0$ will be identified with the number α.

Fig. 1

B. Let the compactum Φ consist of all points of the plane XOY of the form $(x, 0)$, where $0 \leq x \leq 1$, and of the form

$$\left(\frac{k}{2^n}, y\right),$$

where, for any $n = 0, 1, 2, ...$ and any integer k, $0 \leq k \leq 2^n$, the quantity y takes all values from

$$-\frac{1}{2^n}$$

to

$$\frac{1}{2^n}$$

inclusive (see Fig. 1).

C. The required space S is a subset of the topological product $\Gamma \times Y$ of the set Γ, taken in the natural "interval" topology, and the interval $Y = [0, 1]$. For this, the points (x, y) of the plane XOY, for which $0 \leq x < 1$ and $0 \leq y \leq 1$, are numbered by all ordinal numbers α less than $\omega(c)$. The point numbered by the number α will be denoted by (x_α, y_α). Further, in each square Q_α of the product $\Gamma \times Y$, consisting of points of the form $(\alpha x, y)$ and $(\alpha + 1, y)$, where α is fixed and $0 \leq x < 1$ and $0 \leq y \leq 1$, we take the compactum Φ_α, consisting of all such points that

$$(x,\, y - y_\alpha) \in \Phi$$

and at the same time, for $0 < x < 1$, necessarily

$$\frac{y_\alpha}{2} \leq y \leq \frac{y_\alpha + 1}{2}.$$

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* In the expressions "one-dimensional," "zero-dimensional," dimension is understood only in the sense of the small inductive dimension ind, and by the small inductive dimension we mean the inductive dimension in the sense of Urysohn (induction is carried out over points); the inductive dimension Ind in the sense of Čech (induction is carried out over closed sets) we call large. Always $\operatorname{ind} R \leq \operatorname{Ind} R$.

** It is a modification of the well-known space of Lunc ([5]).

The sum of all the compacta Φ_α is the desired space S. It consists of the compacta Φ_α, shifted into the squares Q_α of the product $\Gamma \times Y$, in such a way that on each "horizontal line" $\Gamma \times y$, $0 < y < 1$, there is a continuum of intervals, and they are cut so that they intersect the "lines" $\Gamma \times 0$ and $\Gamma \times 1$ only at the integer points $(\alpha, 0)$ and, respectively, $(\alpha, 1)$.

D. *The space S is normal.*

Indeed, the product $\Gamma \times Y$ is normal (see ([6]), p. 166, Lemma 1), and the space S is closed in it, since it contains all the "vertical" intervals $\alpha \times Y$, $\alpha < \omega(c)$.

E. *No continuum-sized subset A' of the set $A = \{(\alpha, 0)\}$ can be separated by any zero-dimensional set from the set $B = \{(\alpha, 1)\}$.*

Proof. It is enough to prove that the boundary $|U|$ of any canonical** neighborhood U of the set A', whose closure $[U]$ intersects B in a set of cardinality $< c$, contains at least one interval. We shall prove this. Let, for each point $(\alpha, 0) \in A' \subset U$, the number b_α be the greatest of all such numbers y that $\alpha \times [0; y) \subseteq U$. Then $0 \leq b_\alpha \leq 1$ and $(\alpha, b_\alpha) \in |U|$; moreover, since the intersection $[U] \cap B$ has cardinality $< c$, beginning with some α_0, all $b_\alpha < 1$. The set Y' of all numbers b_α, $\alpha \geq \alpha_0$, is either a continuum or is not a continuum.

In the first case there exist two numbers b' and b'', $b' < b''$, which are points of complete accumulation of the set Y'. Hence both the set of all numbers α'', $\alpha'' \geq \alpha_0$, for which

$$b_{\alpha''} > \frac{b' + b''}{2},$$

and the set of all numbers α', $\alpha' \geq \alpha_0$, for which

$$b_{\alpha'} < \frac{b' + b''}{2},$$

are continua, and therefore cofinal with the number $\omega(c)$. In this case necessarily $(\alpha', b_{\alpha'}) \in |V|$, where $V = S \setminus [U]$. Therefore, for every α, $\alpha < \omega(c)$, there exist such a β, $\alpha \leq \beta < \omega(c)$, such a dyadic-rational number x, $0 \leq x < 1$, $\alpha < \beta x$, and such rational numbers r' and r'', $r' < r''$, of the interval Y, that

$$\beta x \times [r'; r''] \subseteq V.$$

The set of all rational intervals $[r'; r'']$ is countable; therefore there exists a continuum set of numbers α to which one and the same interval $[r'; r'']$ is assigned. Hence there is such a sequence of numbers

$$\alpha_1 < \beta_1 x_1 < \alpha_2 < \beta_2 x_2 < \cdots,$$

that

$$b_{\alpha_i} > \frac{b' + b''}{2}$$

and

$$\beta_i x_i \times [r'; r''] \subseteq V,$$

where

$$r' < r'' < \frac{b' + b''}{2}.$$

Then for the limiting number $\gamma = \lim \alpha_i$ we have

$$\gamma \times \left[0; \frac{b' + b''}{2}\right] \subseteq [U]$$

and

$$\gamma \times [r'; r''] \subseteq [V].$$

Therefore the interval $\gamma \times [r'; r'']$ is contained in the boundary $|U|$, as was required to prove.

In the second case, when the set Y' is not a continuum, there will be a number α_1, $\alpha_0 \leq \alpha_1 < \omega(c)$, such that $b_\alpha = b$ for all $\alpha > \alpha_1$. Analogously to the preceding case one can show that $(\alpha x, y) \in U$ as soon as $\alpha_1 \leq \alpha x < \omega(c)$ and $0 \leq y < b$. By construction, the set of all such numbers α that $y_\alpha = b$, and hence $[\alpha; \alpha+1] \times b \subseteq S$, is a continuum. Hence all these intervals $[\alpha; \alpha+1] \times b$ are contained in $[U]$. If $[\alpha; \alpha+1] \times b \subseteq |U|$ for at least one number α, $\alpha \leq \omega(c)$, then everything is proved. In the contrary case, by the construction of the compacta Φ_α, for every such number α that $y_\alpha = b$, there are such rational numbers x and r of the interval $[0; 1]$ that $r > b$ and

$$\alpha x \times [b; r] \subseteq U.$$

But then again there will be a continuum set of such numbers α for which the numbers r coincide with one another. Hence, analogously to the preceding case, one can choose such rational numbers r' and r'' of the interval $[b; r]$ and such a sequence

$$\alpha_1 x_1 < \beta_1 y_1 < \alpha_2 x_2 < \beta_2 y_2 < \cdots,$$

that

$$\alpha_i x_i \times [b; r] \subseteq U,$$

but

$$\beta_i y_i \times [r'; r''] \subseteq V,$$

where $b < r' < r'' < r$. Hence, for the limit number $\gamma = \lim \alpha_i$ we obtain that $\gamma \times [r'; r''] \subseteq [U] \cap [V] = |U|$, which was required to be proved.

§ 2. **The basic space Σ.**

F. By linearly compressing, by means of the mapping

$$f_n(y) = \frac{y}{n} - \frac{y}{n+1} + \frac{1}{n+1},$$

the interval $Y = [0; 1]$ onto the interval $\left[\frac{1}{n+1}; \frac{1}{n}\right]$, $n = 1, 2, \ldots$, we obtain a "compression along the OY-axis" of the product $\Gamma \times Y$, which thereby passes into the product

$$\Gamma \times \left[\frac{1}{n+1}; \frac{1}{n}\right].$$

Under this, the space S is mapped homeomorphically onto closed sets

$$S_n, \qquad S_n \subseteq \Gamma \times \left[\frac{1}{n+1}; \frac{1}{n}\right] \subseteq \Gamma \times Y,$$

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* The sets A and B of the space R are, by definition, *separable by a set C*, if the difference $R \setminus C$ is the sum of such open sets H and G that $A \subseteq H$ and $G \supseteq B$.

** That is, such a neighborhood U that $S \setminus U = [S \setminus [U]]$.

glued pairwise along the "bases"

$$A_n = \left\{ \left(\alpha; \frac{1}{n+1} \right) \right\}$$

and

$$B_{n+1} = \left\{ \left(\alpha, \frac{1}{n+1} \right) \right\}.$$

Adjoining to the set

$$\dot{\Sigma} = \bigcup_n S_n$$

one closed point ξ, with neighborhoods

$$U_n \xi = \xi \cup \left(\dot{\Sigma} \setminus \bigcup_{i \le n} S_i \right), \qquad n = 1, 2, \dots,$$

we obtain the desired space Σ.

G. *The space* Σ *is normal.*

Indeed, the set $\dot{\Sigma}$, being of type F_σ in the normal space $\Gamma \times Y$, is normal. Hence the space $\Sigma = \dot{\Sigma} \cup \xi$ is also normal, since

$$[U_{n+1}\xi] \subset U_n \xi, \qquad n = 1, 2, \dots.$$

H. *The space* Σ *is one-dimensional:* $\operatorname{ind} \Sigma = 1$.

Proof. Since the space Σ contains the interval $0 \times (0;1)$, we have $\operatorname{ind} \Sigma \ge 1$. To prove the reverse inequality, we shall prove the inequality $\operatorname{ind}_x \Sigma \le 1$ for every $x \in \Sigma$.

1°. Since $|U_n \xi| = A_n$ and $\operatorname{ind} A_n = 0$ for every n, it follows that $\operatorname{ind}_\xi \Sigma \le 1$.

2°. Let a point x, $x \ne \xi$, have the form (α, y), where α is a limit number. Then, since the cardinality of the order type $\alpha + 1$ is less than the cardinality of the continuum, there exist numbers $y' \in (y - \varepsilon; y)$ and $y'' \in (y; y + \varepsilon)$ not coinciding with any of the numbers $f_n(y_\lambda)$, $\lambda < \alpha$, $n = 1, 2, \dots$, whatever ε, $0 < \varepsilon < y$, we take. Hence there exists an "arbitrarily small" neighborhood $(\beta t; \alpha t) \times (y'; y'')$ of the point (α, y) in the product $\Gamma \times Y$, where $\xi < \alpha$, and t is a dyadic-irrational number. Therefore the boundary of this neighborhood intersects Σ in a zero-dimensional set. Hence $\operatorname{ind}_x \Sigma \le 1$.

3°. For each of the remaining points x' of the space Σ, $x' \ne \xi$, in the product $\Gamma \times Y$ there is a neighborhood of the form $(\alpha, \alpha + 2) \times (0; 1]$, homeomorphic to an incomplete square. The intersection

$$\Sigma \cap ((\alpha; \alpha + 2] \times (0; 1])$$

is one-dimensional, since it is the sum of a countable number of compacta of the form Φ_α. Hence also $\operatorname{ind}_{x'} \Sigma \le 1$. Thus $\operatorname{ind} \Sigma = 1$, which was required to be proved.

J. *The space* Σ *is not contained in any one-dimensional bicompactum.*

Proof. Suppose that there exists a bicompactum $\widetilde{\Sigma}$ such that

$$\Sigma \subseteq \widetilde{\Sigma}$$

and

$$\operatorname{ind} \widetilde{\Sigma} = 1.$$

Then, by Vedenisov' s theorem XV_1 (^{4}), also

$$\operatorname{Ind} \widetilde{\Sigma} = 1.$$

Hence, for every point x of the bicompactum $\widetilde{\Sigma}$ and every neighborhood $\widetilde{U}_x$ of it, there exist neighborhoods $\widetilde{O}_{nx}$ such that

$$[\widetilde{O}_{nx}] \subseteq \widetilde{O}_{n+1,x} \subseteq \widetilde{U}_x$$

and

$$\operatorname{ind} |\widetilde{O}_{nx}| \leq 0, \qquad n = 1, 2, \dots .$$

But then in the space Σ there would be such neighborhoods $O_n\xi$ of the point ξ that

$$[O_n\xi] \subseteq O_{n+1}\xi \subseteq U_1\xi$$

and

$$\operatorname{ind} |O_n\xi| \leq 0.$$

We shall bring this assertion to a contradiction.

Choose such a number N that

$$[U_N\xi] \subseteq O_1\xi.$$

Then

$$A_N = \left\{ \left(\alpha, \frac{1}{n+1} \right) \right\} = |U_N\xi| \subseteq O_1\xi.$$

Since, by virtue of E, no continuum subset of the set A_{N-1} is separable from the set A_N by any zero-dimensional set, the intersection

$$[O_1\xi] \cap A_{N-1}$$

is a continuum; for otherwise the continuum set

$$A'_{N-1} = A_{N-1} \setminus [O_1\xi]$$

would be separable from A_N by a zero-dimensional set—the boundary of the neighborhood $O_1\xi$. Consequently, again by virtue of E, the set

$$[O_1\xi] \cap A_{N-1}$$

is not separable from A_{N-2} by any

zero-dimensional set. However, $[O_1\xi] \cap A_{N-1} \subseteq O_2\xi$ and $\operatorname{ind}|O_2\xi| = 0$. Hence the intersection $[O_2\xi] \cap A_{N-2}$ is continual and therefore is not separated from A_{N-3} by a zero-dimensional set. Continuing this process, we arrive at the fact that $[O_{N-1}\xi] \cap A_1 \neq \Lambda$, and this contradicts the condition $[O_{N-1}\xi] \subset U_1\xi$, since $\{U_1\xi\} = A_1$, as was required to prove.

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Note: Figure translations are in progress. See original paper for figures.

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