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Abstract

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ON THE REPRESENTATION OF CONTINUOUS FUNCTIONS OF SEVERAL VARIABLES IN THE FORM OF SUPERPOSITIONS OF CONTINUOUS FUNCTIONS OF ONE VARIABLE AND ADDITION

The purpose of this note is to give a brief exposition of the proof of the following theorem:

Theorem. *For every integer $n \geq 2$ there exist real functions $\psi^{pq}(x)$, defined and continuous on the unit interval $E^1 = [0; 1]$, such that every real continuous function $f(x_1, \dots, x_n)$, defined on the n -dimensional unit cube E^n , is representable in the form*

$$f(x_1, \dots, x_n) = \sum_{q=1}^{q=2n+1} \chi_q \left[\sum_{p=1}^n \psi^{pq}(x_p) \right], \quad (1)$$

where the functions $\chi_q(y)$ are real and continuous.

For $n = 3$, putting

$$\varphi_q(x_1, x_2) = \psi^{1q}(x_1) + \psi^{2q}(x_2), \quad h_q(y, x_3) = \chi_q[y + \psi^{3q}(x_3)],$$

we obtain from (1)

$$f(x_1, x_2, x_3) = \sum_{q=1}^7 h_q[\varphi_q(x_1, x_2), x_3], \quad (2)$$

which is a slight strengthening of a result of V. I. Arnol' d ⁽²⁾, who showed that any continuous function of three variables is representable as a sum of nine terms of the same form as the terms entering formula (2), in number seven. The results of my note ⁽¹⁾ do not follow from the new theorem now being communicated in their exact formulations, but their essential content (in the sense of the possibility of representing functions of several variables by superpositions of functions of a smaller number of variables and of approximating them by

superpositions of fixed form from polynomials of one variable and addition) is contained in an obvious way in the new theorem. The method of proof of the new theorem is more elementary than the methods of the works ^(1,2), reducing to direct constructions and estimates. In particular, the need has disappeared to use trees of components of level lines. In fact, however, the constructions used in this note were found by analyzing the constructions used in ^(1,2) and discarding in them details unnecessary for obtaining the final result.

§ 1. **Construction of the functions** ψ^{pq} . The indices p, q, k, i everywhere below run through integer values

$$1 \leq p \leq n, \quad 1 \leq q \leq 2n + 1, \quad k = 1, 2, \dots, \quad 1 \leq i \leq m_k = (9n)^k + 1.$$

In summations and products within these limits the limits will not be indicated.

Consider the segments

$$A_{k,i}^q = \left[\frac{1}{(9n)^k} \left(i - 1 - \frac{q}{3n} \right), \frac{1}{(9n)^k} \left(i - \frac{1}{3n} - \frac{q}{3n} \right) \right].$$

The segments $A_{k,i}^q$ have length $\frac{1}{(9n)^k} \left(1 - \frac{1}{3n} \right)$, and, for fixed k and q , are obtained one from another in passing from i to $i' = i + 1$ by a shift to the right by the distance $\frac{1}{(9n)^k}$, i.e. they are situated not only without overlaps, but with gaps of length $\frac{1}{3n(9n)^k}$, and, up to the presence of these gaps, cover the entire unit interval E^1 . In accordance with this, the cubes

$$S_{k,i_1 \dots i_n}^q = \prod_p A_{k,i_p}^q$$

with edges of length $\frac{1}{(9n)^k}$, for fixed k and q , cover the unit cube E^n , up to the slits separating them, of width $\frac{1}{3n(9n)^k}$. The following lemma is easily verified.

Lemma 1. The system of all cubes $S_{k,i_1 \dots i_n}^q$, with k constant and q and i_1, \dots, i_n variable, covers the unit cube E^n in such a way that each point of E^n is covered at least $n + 1$ times.

By induction on k , the following lemma can be proved.

Lemma 2. The constants $\lambda_{k,i}^{pq}$ and ε_k can be chosen so that the following conditions are fulfilled:

$$1) \lambda_{k,i}^{pq} < \lambda_{k,i+1}^{pq} \leq \lambda_{k,i}^{pq} + \frac{1}{2^k};$$

- 2) $\lambda_{k,i}^{pq} \leq \lambda_{k+1,i'}^{pq} \leq \lambda_{k,i}^{pq} + \varepsilon_k - \varepsilon_{k+1}$, if the segments $A_{k,i}^q$ and $A_{k+1,i'}^q$ intersect;
 3) the segments

$$\Delta_{k,i_1 \dots i_n}^q = \left[\sum_p \lambda_{k,i_p}^{pq}; \sum_p \lambda_{k,i_p}^{pq} + n\varepsilon_k \right]$$

for fixed k and q are pairwise disjoint.

It is easy to see that from 1) and 3) it follows that

$$4) \varepsilon_k < \frac{1}{2^k}.$$

On the basis of the above-mentioned properties of the segments $A_{k,i}^q$ and properties 1), 2), and 4) of the constants $\lambda_{k,i}^{pq}$ and ε_k , the following lemma is proved without much difficulty.

Lemma 3. For fixed p and q , the requirements

$$5) \lambda_{k,i}^{pq} \leq \psi^{pq}(x) \leq \lambda_{k,i}^{pq} + \varepsilon_k \text{ for } x \in A_{k,i}^q$$

uniquely determine a continuous function ψ^{pq} on E^1 .

Remark. It is easy to see that, by construction, the functions ψ^{pq} turn out to be monotonically increasing. This property of theirs could have been included in the formulation of our theorem.

From 5) and 3) it follows that

$$6) \sum_p \psi^{pq}(x_p) \in \Delta_{k,i_1 \dots i_n}^q \text{ when } (x_1, \dots, x_n) \in S_{k,i_1 \dots i_n}^q.$$

§ 2. Construction of the functions χ^q .

Having established the existence of the functions ψ^{pq} and constants $\lambda_{k,i}^{pq}$ and ε_k , possessing properties 1)–6), we pass to the proof of the main theorem. The required functions $\chi^q(y)$ will be constructed in the form

$$\chi^q = \lim_{r \rightarrow \infty} \chi_r^q,$$

where $\chi_0^r \equiv 0$, and χ_r^q for $r > 0$ will be defined by induction on r , simultaneously with natural numbers k_r .

We shall use the notation

$$f_r(x_1, \dots, x_n) = \sum_q \chi_r^q \left[\sum_p \psi^{pq}(x_p) \right], \quad (3)$$

$$M_r = \sup_{E^n} |f - f_r|. \quad (4)$$

Obviously,

$$f_0 \equiv 0, \quad M_0 = \sup_{E^n} |f|.$$

Suppose that the continuous functions χ_{r-1}^q and the number k_{r-1} have already been determined. Thus the continuous function f_{r-1} is also determined on E^n . Since, as $k \rightarrow \infty$, the diameters of the cubes $S_{k,i_1 \dots i_n}^q$ tend to zero, one can choose k_r so large that the oscillation of the difference $f - f_{r-1}$ on any $S_{k_r, i_1 \dots i_n}^q$ does not exceed

$$\frac{1}{2n+2} M_{r-1}.$$

Let $\xi_{k,i}^q$ be arbitrary points of the corresponding segments $A_{k,i}^q$. On the segment $\Delta_{k,i_1 \dots i_n}^q$ put

$$\chi_r^q(y) = \chi_{r-1}^q(y) + \frac{1}{n+1} [f(\xi_{k,i_1}^q, \dots, \xi_{k,i_n}^q) - f_{r-1}(\xi_{k,i_1}^q, \dots, \xi_{k,i_n}^q)]. \quad (5)$$

It is obvious that the values of the functions χ_r^q fixed in this way are subject to the inequality

$$|\chi_r^q(y) - \chi_{r-1}^q(y)| \leq \frac{1}{n+1} M_{r-1}. \quad (6)$$

Outside the segments $\Delta_{k,i_1 \dots i_n}^q$ we complete the definition of the function χ_r^q arbitrarily, but preserving this same inequality (6) and continuity.

We now estimate $f - f_r$ at an arbitrary point (x_1, \dots, x_n) of E^n . It is obvious that

$$f(x_1, \dots, x_n) - f_r(x_1, \dots, x_n) = f(x_1, \dots, x_n) - f_{r-1}(x_1, \dots, x_n) - \sum_q \left\{ \chi_r^q \left[\sum_p \psi^{pq}(x_p) \right] - \chi_{r-1}^q \left[\sum_p \psi^{pq}(x_p) \right] \right\}. \quad (7)$$

We represent the sum \sum_q in (7) in the form $\sum' + \sum''$, where the sum \sum' extends over those $n+1$ values of q for which the point (x_1, \dots, x_n) belongs to one of the cubes $S_{k,i_1 \dots i_n}^q$ (such values exist by Lemma 1), while the sum \sum'' extends over the remaining n values of q .

For each term of \sum' we obtain, by virtue of (5),

$$\begin{aligned} & \chi_r^q \left[\sum_p \psi^{pq}(x_p) \right] - \chi_{r-1}^q \left[\sum_p \psi^{pq}(x_p) \right] = \\ & \frac{1}{n+1} [f(\xi_{k,i_1}^q, \dots, \xi_{k,i_n}^q) - f_{r-1}(\xi_{k,i_1}^q, \dots, \xi_{k,i_n}^q)] = \\ & \frac{1}{n+1} [f(x_1, \dots, x_n) - f_{r-1}(x_1, \dots, x_n)] + \frac{\omega^q}{n+1}, \end{aligned} \quad (8)$$

where

$$|\omega^q| \leq \frac{1}{2n+2} M_{r-1}. \quad (9)$$

The terms from \sum'' are estimated by means of (6). From (5), together with (8), (9), and (6), we obtain

$$\begin{aligned} |f - f_r| &= \left| \frac{1}{n+1} \sum' \omega^q + \sum'' (\chi_r^q - \chi_{r-1}^q) \right| \leq \\ & \leq \frac{1}{2n+2} M_{r-1} + \frac{n}{n+1} M_{r-1} = \frac{2n+1}{2n+2} M_{r-1}. \end{aligned} \quad (10)$$

Since inequality (10) is valid at every point $(x_1, \dots, x_n) \in E^n$, it follows that

$$\begin{aligned} M_r &\leq \frac{2n+1}{2n+2} M_{r-1}, \\ M_r &\leq \left(\frac{2n+1}{2n+2} \right)^r M_0. \end{aligned} \quad (11)$$

It follows from (6) and (11) that the differences $\chi_r^q - \chi_{r-1}^q$ do not exceed in absolute value the corresponding terms of the absolutely convergent series

$$\sum_r \frac{1}{n+1} M_{r-1}.$$

Therefore the functions χ_r^q , as $r \rightarrow \infty$, converge uniformly to continuous limiting functions χ^q .

From relations (3) and (4) and estimate (11), by passing to the limit as $r \rightarrow \infty$, we obtain equality (1), which completes the proof of the theorem.

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CITED LITERATURE

¹ A. N. Kolmogorov, DAN, **108**, No. 2 (1956).

² V. I. Arnold, DAN, **114**, No. 4 (1957).

Note: Figure translations are in progress. See original paper for figures.

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