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Abstract

Full Text

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ON SYSTEMS OF QUASILINEAR EQUATIONS

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A characteristic property of solutions of quasilinear equations of hyperbolic type is the formation of a discontinuity in the solution even for smooth initial data. In connection with this, for systems of quasilinear equations written in the form (2), the concept of a generalized or weak solution of the system is introduced ([1],[2]).

Here we consider the question of the possibility of representing an arbitrary system of quasilinear equations (1) in the form of conservation laws (2). The case of the equations of hydrodynamics is considered in detail; for it all possible conservation laws are found. It turns out that the number of conservation laws is greater than the number of unknown functions. In connection with this one can indicate several generalized solutions of the Cauchy problem for the equations of hydrodynamics.

Consider a system of quasilinear equations:

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{n} a_{ij}\frac{\partial u_j}{\partial x} = b_i \quad (i = 1, 2, \dots, n), \tag{1}$$

where the coefficients a_{ij}, b_i are analytic functions of the variables u_j, t, x. With respect to the system (1) we assume that it is hyperbolic in the region under consideration of variation of the variables u_j, t, x.

Suppose that the system (1) can be reduced to the form

$$\frac{\partial \varphi_i(u_j, t, x)}{\partial t} + \frac{\partial \psi_i(u_j, t, x)}{\partial x} = f_i(u_j, t, x) \quad (i = 1, 2, \dots, n). \tag{2}$$

Then the functions $u_j(t, x)$, taking prescribed values on the initial curve and satisfying the integral relations

$$\oint_C \varphi_i(u_j, t, x)\, dx - \psi_i(u_j, t, x)\, dt = \iint_G f_i(u_j, t, x)\, dt\, dx \quad (i = 1, 2, \dots, n) \tag{3}$$

(where C is any contour bounding the domain G of the variables t, x), are called a **generalized solution of the Cauchy problem for the system** (1).

It is obvious that the inverse passage from the relations (3) to (2) is also possible, provided only that $u_j(t,x)$ are smooth functions.

Definition 1. If the equation

$$\frac{\partial \varphi(u_j,t,x)}{\partial t} + \frac{\partial \psi(u_j,t,x)}{\partial x} = f(u_j,t,x) \tag{4}$$

is a consequence of the system (1) for arbitrary smooth solutions $u_j(t,x)$, then we shall call it a **conservation law of the system** (1).

Definition 2. If system (1) has at least n independent conservation laws, then we shall call it **conservative**; otherwise we shall call system (1) **nonconservative**.

Definition 3. A system (1) that has exactly n independent conservation laws will be called **conservatively closed**. We shall call system (1) **completely nonconservative** in the case when it has not a single conservation law.

In these definitions the notion of independence of conservation laws occurs. By this we mean the following: if system (1) has K functions $\varphi_1(u_j,t,x), \varphi_2(u_j,t,x), \dots, \varphi_k(u_j,t,x)$ entering equation (4), then these conservation laws (4) are called **independent** if, for any fixed values of the variables t, x from the domain under consideration, the system of functions $1, \varphi_1(u_j,t,x), \varphi_2(u_j,t,x), \dots, \varphi_k(u_j,t,x)$ is linearly independent.

The following assertions hold:

1. *System (1) for $n \le 2$ is always conservative.*
2. *System (1) for $n \ge 3$ is, as a rule, a completely nonconservative system.*
3. *The property of the system of quasilinear equations (1) of being conservative is preserved under any nondegenerate transformation of the dependent k independent variables. Thus, the property of conservativity is a fundamental property of systems of quasilinear equations, invariant under any transformation of variables.*

Of these assertions, the first is obvious. To prove the other two, multiply each equation of system (1) by $k_i(u_j,t,x)$, sum the results over i, and require that the equation obtained have the form of the conservation law (4). We obtain the system of equations

$$k_i = \frac{\partial \varphi}{\partial u_i}; \qquad \sum_{j=1}^{n} k_j a_{ji} = \frac{\partial \psi}{\partial u_i} \quad (i = 1, 2, \dots, n) \tag{5}$$

and the equation for determining the right-hand side of the conservation law (4)

$$f(u_j, t, x) = \sum_{j=1}^{n} k_j b_j - \varphi'_t - \psi'_x.$$

Eliminating from system (5) the unknowns $k_i(u_j, t, x)$, we obtain

$$\frac{\partial \psi}{\partial u_i} = \sum_{j=1}^{n} a_{ji} \frac{\partial \varphi}{\partial u_j} \quad (i = 1, 2, \dots, n). \tag{6}$$

System (6) is a linear homogeneous system of first-order partial differential equations with respect to the unknown functions $\psi(u_j, t, x)$, $\varphi(u_j, t, x)$, depending parametrically on the variables t and x. Since multiplying solutions of system (6) by arbitrary functions of the variables t, x does not give new independent conservation laws of system (1), we shall regard t and x in equations (6) as fixed.

Let us prove assertion 2. In the case $n \geq 3$, system (6) is overdetermined. If system (6) has nontrivial solutions $\varphi(u_j, t, x)$, $\psi(u_j, t, x)$, then these solutions must also satisfy all the equations of the prolonged system obtained from system (6) by differentiation with respect to the variables u_j ([3]). Treating the derivatives of the functions ψ, φ as algebraic unknowns and carrying out all independent differentiations of equations (6), we find that, starting from some point, the number of equations becomes greater than the number of unknowns. If, moreover, all equations obtained by differentiating system (6) are algebraically independent —and this should occur in the case of arbitrary coefficients $a_{ji}(u_j, t, x)$—then the system

(6) will have only trivial solutions. It may happen, however, that part of the equations of the prolonged system will be an algebraic consequence of system (6) and its prolongations, so that the number of unknown derivatives will always be greater than the number of independent equations of the prolonged system. In this case system (6) may have nontrivial solutions, depending both on a number of arbitrary constants and on certain arbitrary functions of the variables u_j.

Thus, assertion 2 is proved. It follows from this that, beginning with $n = 3$, there exist completely nonconservative, conservative-closed, and conservative systems. This circumstance justifies the above classification of systems of quasilinear equations according to the property of conservativity.

The existence of nonconservative systems is an essential property of systems of quasilinear equations, distinguishing them from linear systems, since every linear system of equations is, obviously, conservative.

It also follows from what has been said that, for nonconservative systems of quasilinear equations, the notion of a generalized solution as a solution satisfying a number of integral relations loses its meaning.

As for assertion 3, it is also obvious, since the property of system (6) of having solutions does not depend on a transformation of the variables u_j and the parameters t, x.

A special place among systems of quasilinear equations is occupied by the equations of hydrodynamics. We set ourselves the task of determining the class of conservation laws for the equations of hydrodynamics of one-dimensional flow of a compressible fluid.

Writing them in Euler variables, we shall have:

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v}{\partial x}=0; \qquad \rho\frac{\partial v}{\partial t}+\rho v\frac{\partial v}{\partial x}+\frac{\partial p}{\partial x}=0;$$

$$p=p(\rho,\varepsilon); \qquad \frac{\partial\varepsilon}{\partial t}+v\frac{\partial\varepsilon}{\partial x}-\frac{p}{\rho^2}\left[\frac{\partial\rho}{\partial t}+v\frac{\partial\rho}{\partial x}\right]=0, \tag{7}$$

where ρ is density; v is velocity; ε is internal energy, and $p=p(\rho,\varepsilon)$ is pressure, given as an arbitrary function of the density and internal energy of the gas. Writing, for equations (7), the corresponding system (6), we obtain:

$$\frac{\partial\psi}{\partial\rho}=v\frac{\partial\varphi}{\partial\rho}+\frac{p'_\rho}{\rho}\frac{\partial\varphi}{\partial v}; \qquad \frac{\partial\psi}{\partial v}=\rho\frac{\partial\varphi}{\partial\rho}+v\frac{\partial\varphi}{\partial v}+\frac{p}{\rho}\frac{\partial\varphi}{\partial\varepsilon};$$

$$\frac{\partial\psi}{\partial\varepsilon}=\frac{p'_\varepsilon}{\rho}\frac{\partial\varphi}{\partial v}+v\frac{\partial\varphi}{\partial\varepsilon}. \tag{8}$$

Our task consists in finding the general solution of system (8). Eliminating $\psi(\rho,v,\varepsilon)$ from (8), we obtain

$$\frac{\partial}{\partial\rho}\left[\rho\frac{\partial\varphi}{\partial\rho}+\frac{p}{\rho}\frac{\partial\varphi}{\partial\varepsilon}-\varphi\right]=\frac{p'_\rho}{\rho}\frac{\partial^2\varphi}{\partial v^2}; \qquad \frac{\partial}{\partial\varepsilon}\left[\rho\frac{\partial\varphi}{\partial\rho}+\frac{p}{\rho}\frac{\partial\varphi}{\partial\varepsilon}-\varphi\right]=\frac{p'_\varepsilon}{\rho}\frac{\partial^2\varphi}{\partial v^2};$$

$$p'_\varepsilon\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial v}\right]=p'_\rho\frac{\partial}{\partial\varepsilon}\left[\frac{1}{\rho}\frac{\partial\varphi}{\partial v}\right]. \tag{9}$$

From (9) we obtain:

$$\varphi(\rho,v,\varepsilon)=\rho\Phi(\rho,v)+F(\rho,\varepsilon); \tag{10}$$

$$\rho\frac{\partial\varphi}{\partial\rho}+\frac{p}{\rho}\frac{\partial\varphi}{\partial\varepsilon}-\varphi=C(\rho,v), \tag{11}$$

where Φ, F, C are arbitrary functions of their arguments.

Substituting (10) into (11), we find:

$$\Phi(p,v)=\Phi_1(v);\qquad C(p,v)=C_1(p);\qquad \varphi(\rho,v,\varepsilon)=\rho\Phi_1(v)+F(\rho,\varepsilon). \quad (12)$$

Now we finally find $\varphi(\rho,v,\varepsilon)$ and $\psi(\rho,v,\varepsilon)$ from (12), (9), and (8):

$$\begin{aligned}\varphi(\rho,v,\varepsilon)&=C_1\rho+C_2\rho v+C_3\left(\rho\varepsilon+\rho\frac{v^2}{2}\right)+C_4\rho f(S);\\ \psi(\rho,v,\varepsilon)&=C_1\rho v+C_2(\rho v^2+p)+C_3\left(\rho v\varepsilon+pv+\rho\frac{v^3}{2}\right)+C_4\rho v f(S),\end{aligned} \quad (13)$$

where $f=\text{const}$ is an integral of the equation

$$d\varepsilon-\frac{p}{\rho^2}\,d\rho=0.$$

The general solution obtained for the system of equations (8) expresses, as is not difficult to see, the conservation laws of mass, momentum, energy, and entropy of a gas, well known in hydrodynamics. From this point of view, all four terms in (13) could have been written down in advance; however, a priori it was unclear whether they express all possible conservation laws for the equations of hydrodynamics. Here we thus obtain the conclusion that the system of equations of hydrodynamics is a conservatively nonclosed system.

It is clear from what has been set forth that various generalized solutions of the equations of hydrodynamics can be constructed that satisfy the same initial data.

Discontinuous solutions of systems of quasilinear equations of hyperbolic type can be regarded as limiting solutions of these systems with an introduced "viscosity" in the form of second-order derivatives with a small parameter in front of them, as the latter tends to zero. Since for conservatively nonclosed systems there exist various generalized solutions, by introducing different "viscosities" we can also obtain different limiting solutions.

From the fact that the system of equations of hydrodynamics is also a conservatively nonclosed system, it follows that the method, widely used in computations of hydrodynamic problems, of introducing "viscosities" requires some caution. It is not difficult to give an example of a "viscosity" for which physically incorrect conservation laws will be satisfied.

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Note: Figure translations are in progress. See original paper for figures.