MATHEMATICS

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Abstract

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MATHEMATICS

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CONVERGENCE OF THE LAGRANGE INTERPOLATION PROCESS CONSTRUCTED FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND FUNCTIONS OF BOUNDED VARIATION

(Presented by Academician V. I. Smirnov on 25 VII 1956)

Let a triangular matrix of nodes be given

$$\begin{array}{cccc} & x_1^{(1)} & & \\ & x_1^{(2)} \quad x_2^{(2)} & & \\ & \cdots & & \\ x_1^{(n)} & x_2^{(n)} & \ldots & x_n^{(n)} \\ & \cdots & & \end{array} \tag{1}$$

$$-1 \leq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leq 1 \quad (n = 1, 2, \ldots).$$

Introduce the following notation:

$$\Delta_n = \max_{k=1,2,\ldots,n-1} \left[\left(x_{k+1}^{(n)} - x_k^{(n)} \right) \right].$$

Denote by $\mathfrak{S}(a, b)$ the number of nodes in the n-th row of the matrix (1) satisfying the inequalities $a \leq x_j^{(n)} \leq b$ (or $b \leq x_j^{(n)} \leq a$). Let

$$\omega_n(x) = \prod_{j=1}^{n} \left(x - x_j^{(n)} \right) \quad (n = 1, 2, \ldots);$$

$$l_j^{(n)}(x) = \frac{\omega_n(x)}{\left(x - x_j^{(n)} \right) \omega_n' \left(x_j^{(n)} \right)} \quad (j = 1, 2, \ldots, n;\ n = 1, 2, \ldots);$$

$$\lambda_j^{(n)}(x) = \sum_{k=j}^{n} l_k^{(n)}(x).$$

For any function $f(x)$ defined on the segment $[-1,1]$, one can construct the Lagrange interpolation polynomial $L_n(f,x)$

$$L_n(f,x) = \sum_{k=1}^{n} f\left(x_k^{(n)}\right) l_k^{(n)}(x).$$

Recently V. I. Krylov ([1]) proved the theorem:

In order that, for every absolutely continuous function $f(x)$, the relation

$$L_n(f,x) \to f(x), \qquad n \to \infty, \tag{2}$$

hold uniformly on the segment $[-1,1]$, it is necessary and sufficient that there exist a number A such that, for all n, j, and $x \in [-1,1]$, the inequality

$$\left|\lambda_j^{(n)}(x)\right| \le A \tag{3}$$

be satisfied.

Han ([2]) and S. M. Lozinskii ([3]) proved the theorem:

For the relation (2) to hold at the point x for every function $f(x)$ of bounded variation on $[-1,1]$ and continuous at the point $x \in [-1,1]$, it is necessary and sufficient that the following conditions hold:

1) At the point x the inequality

$$\left|\lambda_j^{(n)}(x)\right| \le A(x), \qquad j = 1,2,\dots,n, \quad n = 1,2,\dots, \tag{4}$$

   holds, where the finite number $A(x)$ depends only on x.

2) If $t \in [-1,1]$ and is distinct from the nodes $\{x_k^{(n)}\}_{k=1}^{n}$, $n = 1,2,\dots$, then the equalities

$$\lim_{n\to\infty} \sum_{x_k^{(n)}<t} l_k^{(n)}(x) = 0 \qquad \text{for } t < x,$$

$$\lim_{n\to\infty} \sum_{x_k^{(n)}>t} l_k^{(n)}(x) = 0 \qquad \text{for } t > x. \tag{5}$$

Using the indicated theorems, V. I. Krylov ([4]) established the convergence of the Lagrange interpolation process constructed at the nodes of P. L. Chebyshev

$$x_k^{(n)} = \cos\frac{2n-2k+1}{2n}, \qquad k = 1,2,\dots; \quad n = 1,2,\dots,$$

in the case when $f(x)$ is absolutely continuous or has bounded variation and is continuous at the given point of $[-1,1]$.

In the present note the theorems of V. I. Krylov ([4]) are extended to a certain broad class of node matrices (1), of which the nodes of P. L. Chebyshev are a particular case.

Theorem 1. Let the matrix (1) satisfy the conditions:

A. At each point $x \in [-1,1]$ the inequalities hold:

$$\text{if } x_k^{(n)} < x_{k+1}^{(n)} \le x \qquad \left|l_k^{(n)}(x)\right| \le \left|l_{k+1}^{(n)}(x)\right| \qquad (n = n_0, n_0+1, \ldots);$$

$$\text{if } x \le x_k^{(n)} < x_{k+1}^{(n)} \qquad \left|l_k^{(n)}(x)\right| \ge \left|l_{k+1}^{(n)}(x)\right| \qquad (n = n_0, n_0+1, \ldots).$$

B. There exists a nonnegative decreasing function $\varphi(h)$, satisfying the condition $\varphi(h) \to 0$ as $h \to \infty$, such that, if $\mathcal{E}(x, x_k^{(n)}) = h$, then

$$\left|l_k^{(n)}(x)\right| \le N\varphi(h) \qquad (n = n_0, n_0+1, \ldots), \quad -1 \le x \le 1,$$

where N is a finite nonnegative number.

Then the Lagrange interpolation process $L_n(f,x)$, constructed for an absolutely continuous function $f(x)$, converges uniformly on $[-1,1]$ to $f(x)$.

Proof. Obviously, it suffices to prove the validity of inequality (3). Let $x_p^{(n)} < x < x_{p+1}^{(n)}$, and let $j < p$*; then

$$\left|\lambda_j^{(n)}(x)\right| \le \left|\sum_{k=j}^{p} l_k\right| + \left|\sum_{k=p+1}^{n} l_k\right|, \qquad l_k = l_k^{(n)}(x). \tag{6}$$

Since $\sum_{k=j}^{p} l_k$ is estimated in exactly the same way as $\sum_{k=p+1}^{n} l_k$, it is enough to consider only $\sum_{k=p+1}^{n} l_k$. Obviously,

$$\left|\sum_{k=p+1}^{n} l_k\right| \le |l_{p+1} + l_{p+2}| + |l_{p+3} + l_{p+4}| + \cdots \tag{7}$$

* In the opposite case the arguments are simplified.

It is easy to see that, for $k > p+1$, $\operatorname{sign} l_k(x) = -\operatorname{sign} l_{k+1}(x)$. Therefore, in accordance with property A of the matrix (1), inequality (7) can be written in the form

$$\left|\lambda_{p+1}^{(n)}\right| \le (|l_{p+1}| - |l_{p+2}|) + (|l_{p+3}| - |l_{p+4}|) + \cdots \le$$

$$\le |l_{p+1}| - (|l_{p+2}| - |l_{p+3}|) - \cdots . \tag{8}$$

If we again use property A of the matrix (1), then from (8) it follows that

$$\left|\lambda_{p+1}^{(n)}(x)\right| \le |l_{p+1}|. \tag{9}$$

Taking into account property B of the matrix (1), by virtue of (9) one can obtain the inequality

$$\left|\lambda_{p+1}^{(n)}(x)\right| \le N\varphi(1). \tag{10}$$

Thus, from (6) and (9) it follows that

$$\left|\lambda_{j}^{(n)}(x)\right| \le 2N\varphi(1) < \infty, \qquad x \in [-1,1], \qquad j = 1,2,\dots,n; \quad n = n_0,\ n_0{+}1,\dots$$

Theorem 1 is proved.

Remark. Theorem 1 can be formulated in a local form. In this case it is sufficient that conditions A and B be satisfied only at the given point of $[-1,1]$.

Theorem 2. Let the n-th row of the matrix (1) be composed of the roots of the Jacobi polynomials

$$Y_n^{(\alpha_n,\beta_n)}(x),$$

where

$$-1 \le \alpha_n,\ \beta_n \le -\lambda < 0, \qquad n = 1,2,\dots,*,$$

where λ is an arbitrarily small positive number.

Then, for any absolutely continuous function $f(x)$, relation (2) holds uniformly on the segment $[-1,1]$.

If the matrix (1) is composed of the roots of the Legendre polynomials ($\alpha_n = \beta_n = 0$) and $f(x)$ is absolutely continuous, then relation (2) holds at every point of the interval $(-1,1)$. The convergence is uniform on any segment of the form $[-1+\varepsilon,\ 1-\varepsilon]$, $0<\varepsilon<1$.

Proof. In my note ([5]) it was proved that the matrix (1), composed of the roots of the polynomials

$$Y_n^{(\alpha_n,\beta_n)}(x),$$

where

$$-1 \le \alpha_n,\ \beta_n \le -\lambda < 0, \qquad n = 1,2,\dots,$$

satisfies conditions A and B of Theorem 1. Therefore the first part of Theorem 2 follows directly from Theorem 1. The second part of Theorem 2 follows from the local form of Theorem 1 and from the fact that the matrix (1), composed of the roots of the Legendre polynomials, satisfies conditions A and B of Theorem 1 in the local form ([5]).

Theorem 3. Let the matrix (1) satisfy conditions A and B of Theorem 1.

Then, if the function $f(x)$ has bounded variation on $[-1, 1]$, relation (2) holds at all points of continuity of $f(x)$.

To prove Theorem 3 it is sufficient to prove the validity of the equalities (5). We shall prove the second of the equalities (5); the first equality is proved in the same way.

Let $x_s < t < x_{s+1}$; then from inequality (9) it follows that

$$\left| \sum_{x_k^{(n)} > t} l_k(x) \right| = \left| \lambda_{s+1}^{(n)}(x) \right| \leq |l_{s+1}|.$$

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* By definition $\quad Y_n^{(-1,-1)}(x) = \int_{-1}^{x} P_{n-1}(t)\, dt, \quad$ where $P_n(t)$ is the Legendre polynomial of degree n.

But, according to condition B,

$$|l_{s+1}| \leq N\varphi\left(\frac{t-x}{\Delta_n}\right), \qquad t > x.$$

Therefore

$$\left| \sum_{x_k^{(n)} > t} l_k(x) \right| \leq N\varphi\left(\frac{t-x}{\Delta_n}\right), \qquad t > x. \tag{11}$$

Since $\Delta_n \to 0$ ([6]), it follows from inequality (11) and the property of the function $\varphi(h)$ that (5) is fulfilled.

Theorem 4. Let the n-th row of the matrix (1) be composed of the zeros of the Jacobi polynomials $Y_n^{(\alpha_n, \beta_n)}(x)$, $-1 \leq \alpha_n,\ \beta_n \leq 0,\ n = 1, 2, \ldots$.

Then, if the function $f(x)$ has bounded variation on $[-1, 1]$, relation (2) is fulfilled at all points of continuity of $f(x)$ in $(-1, 1)$.

This theorem follows directly from Theorem 3 and from the results ([5]).

It is clear that the results of V. I. Krylov ([4]) are special cases of Theorems 2 and 4, when $\alpha_n = \beta_n = -1/2$.

In conclusion I note that the method of proof presented here is of a more elementary character than the method of proof in ([4]).

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CITED LITERATURE

[1] V. I. Krylov, DAN, **105**, No. 2 (1955).
[2] H. Hahn, Math. Zs., **1**, 115 (1918).
[3] S. M. Lozinskii, Uchen. zap. Leningr. univ., No. 55, 84 (1940).
[4] V. I. Krylov, DAN, **107**, No. 3 (1956).
[5] D. L. Berman, DAN, **60**, No. 3 (1948).
[6] P. Erdös, P. Turán, Ann. of Math., **39**, No. 4, 703 (1938).

Note: Figure translations are in progress. See original paper for figures.