SOME PLANE PROBLEMS IN THE THEORY OF THERMAL WAVES

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Abstract

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SOME PLANE PROBLEMS IN THE THEORY OF THERMAL WAVES

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In this note we examine problems on thermal waves in plates whose thickness h is sufficiently small; the temperature is assumed constant through the thickness; the specific heat c, density ρ, and thermal-conductivity coefficient λ are constants; the coordinate axes x, y are placed in the middle plane. Here two types of boundary conditions on the planes $z = \pm h/2$, which we shall call respectively the upper and lower planes, are considered: 1) both planes are adiabatic boundaries; this case coincides with the plane problem on thermal waves in an unbounded cylinder; 2) on both planes heat exchange with the medium occurs, described by boundary conditions of the third kind. For both cases the problem on thermal waves has much in common with problems on the bending of plates on an elastic foundation.

1. The plane problem of the theory of thermal waves under conditions of the first type is described by the differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{a}\frac{\partial T}{\partial t} + F(x, y)\sin \omega t, \tag{1}$$

where $T(x, y, t)$ is the temperature measured from some mean temperature T_0; $F(x, y)$ is the law of distribution of heat sources, which, to shorten the notation and without loss of generality, we shall assume to be only in the phase $\sin \omega t$ (ω is the circular frequency, $a = \lambda/c\rho$); moreover, we shall assume that in the boundary conditions along the contour of the plate (or respectively along the lateral surface of the cylinder) the right-hand side represents some function of the contour arc multiplied by $\sin \omega t$.

If in (1) we put $F(x, y) = 0$, then

$$T(x, y, t) = \varphi(x, y)\cos \omega t + \varphi_1(x, y)\sin \omega t, \tag{2}$$

where $\varphi(x, y)$ and $\varphi_1(x, y)$ satisfy the differential equation

$$\nabla^2\nabla^2\varphi + \left(\frac{\omega}{a}\right)^2 \varphi = 0 \tag{3}$$

and, moreover,

$$\varphi_1 = \frac{a}{\omega}\nabla^2\varphi. \tag{4}$$

Let us pass to dimensionless coordinates; in what follows we adopt the polar coordinate system (ξ, θ), where $\xi = r/l$, $r = \sqrt{x^2 + y^2}$, $l = \sqrt{a/\omega}$.

It is sufficient for us to find $\varphi(x, y)$; after that (2) and (4) give the complete solution of the problem. Since φ satisfies equation (3), the solution of which has been rather well studied in the theory of bending of plates on an elastic foundation, the presentation in many cases is, as already noted, simplified.

Consider the problem of a point source of intensity $Q \sin \omega t$, applied to an unbounded plate; we take the point of application of the source as the origin of coordinates; from what was said above it follows that φ will be ···

differ only by a constant factor from Hertz’ s solution for a force applied to a floating plate; it is easy to show that

$$\varphi(\xi) = \frac{Q}{4\lambda h} f_0(\xi), \tag{5}$$

$$T(\xi, t) = \frac{Q}{4\lambda h} [f_0(\xi)\cos\omega t - g_0(\xi)\sin\omega t], \tag{6}$$

where $f_n(\xi) + i g_n(\xi) = H^{(1)}(\xi\sqrt{i})$; moreover, we denote $u_n(\xi) + i v_n(\xi) = I_n(\xi\sqrt{i})$.

Integrating (5), we easily find $\varphi(\xi, \theta)$, and then T, for cases when the sources are distributed over circular regions. Thus, for example, if sources $q \sin \omega t$ are uniformly distributed over a circle of radius $r_a = \alpha l$, then for $\xi \leqslant \alpha$

$$T = \frac{\pi q l \alpha}{2\lambda h}\{[f_0(\alpha)u_0(\xi) - g_0(\alpha)v_0(\xi)]\cos\omega t -$$

$$-[f_0(\alpha)v_0(\xi) + g_0(\alpha)u_0(\xi)]\sin\omega t\},$$

for $\xi \geqslant \alpha$

$$T = \frac{\pi q l \alpha}{2\lambda h}\{[u_0(\alpha)f_0(\xi) - v_0(\alpha)g_0(\xi)]\cos\omega t - \tag{7}$$

$$-[v_0(\alpha)f_0(\xi) + u_0(\alpha)g_0(\xi)]\sin\omega t\}.$$

The functions $\varphi(\xi, \theta)$ for the problem of sources $q\xi^n \cos n\theta$, distributed over a circular ring or a circle, can be obtained from the solutions given in (1), taking

q to be the intensity of the sources and introducing an additional factor $D/\lambda h l_1^2$, where D is the cylindrical stiffness and l_1 is the characteristic of flexibility.

Applying the usual techniques (set forth, for example, in (1)), we find, by adding to the obtained solutions integrals of the homogeneous equation of the form

$$T_k = [b_1 u_n(\xi) + b_2 v_n(\xi)] \cos \omega t + [b_1 v_n(\xi) - b_2 u_n(\xi)] \sin \omega t +$$

$$+ [b_3 f_n(\xi) + b_4 g_n(\xi)] \cos \omega t + [b_3 g_n(\xi) - b_4 f_n(\xi)] \sin \omega t, \tag{8}$$

the temperature in a circular plate or an infinite plate with a hole.

In order to simplify the calculations in determining the coefficients b_1, b_2, b_3, b_4, in some cases it is convenient to use the method of initial parameters. We shall denote functions possessing the unit-matrix property by $Y_1^*(\alpha, \xi)$, $Y_2^*(\alpha, \xi)$, $Y_3^*(\alpha, \xi)$, $Y_4^*(\alpha, \xi)$. These functions can be obtained by means of formula (2) from (1); thus, for example:

$$Y_1^*(\alpha, \xi) = -Y_3(\alpha_1, \xi) \cos \omega t + Y_1(\alpha, \xi) \sin \omega t, \tag{9}$$

where Y_1, Y_3 are the corresponding functions for a circular plate on an elastic foundation, which are given in (1). Hence

$$Y_1(\alpha, \xi) = \frac{\pi\alpha}{2} [g_0'(\alpha) u_0(\xi) + f_0'(\alpha) v_0(\xi) - v_0'(\alpha) f_0(\xi) - u_0'(\alpha) g_0(\xi)],$$

$$Y_3(\alpha, \xi) = \frac{\pi\alpha}{2} [-f_0'(\alpha) u_0(\xi) + g_0'(\alpha) v_0(\xi) + u_0'(\alpha) f_0(\xi) - v_0'(\alpha) g_0(\xi)].$$

Let us consider the plane problem of heat waves for a plate with adiabatic bases or a cylinder, assuming that the cross-section is bounded by a simple contour C having continuous curvature. Suppose that the temperature on the contour is

$$T(\sigma) = F_1(\sigma) \sin \omega t + F_2(\sigma) \cos \omega t. \tag{10}$$

Here and below, σ is the s-arc coordinate of a point of the contour. Let $\eta = r/l$ denote the reduced distance between a point of the contour and a point of the domain. We shall seek the solution in the form

$$T(x, y; t) = \int_C \mu_1(s) K_1^* \, ds + \int_C \mu_2(s) K_2^* \, ds, \tag{11}$$

where

$$K_1^* = [f_0'(\eta)\sin\omega t - g_0'(\eta)\cos\omega t]\cos(n_s, r_1),$$
$$K_2^* = [g_0'(\eta)\sin\omega t + f_0'(\eta)\cos\omega t]\cos(n_s, r_1), \tag{12}$$

μ_1, μ_2 are some as yet unknown singularity densities.

Since K_1^*, K_2^* satisfy the differential equation of plane heat waves, the solution represented by formula (11) satisfies the differential equation of heat conduction. It is necessary to find the functions μ_1, μ_2 so as to satisfy the boundary conditions. We note that as $\eta \to 0$ the function f_0' remains bounded, while the function $g_0' \to 2/\pi\eta$; therefore, using the known results of the theory of logarithmic potential, as the point of the domain tends to a point of the contour we obtain the following two integral equations:

$$2\mu_1(\sigma) + \int_C \mu_1(s)K_{11}^*(\sigma, s)\,ds + \int_C \mu_2(s)K_{21}^*(\sigma, s)\,ds = F_2(\sigma),$$

$$-2\mu_2(\sigma) + \int_C \mu_1(s)K_{12}^*(\sigma, s)\,ds + \int_C \mu_2(s)K_{22}^*(\sigma, s)\,ds = F_1(\sigma), \tag{13}$$

where

$$K_{11}^* = -K_{22}^* = g_0(\eta_{\sigma,s})\cos(n_s, r_1),$$
$$K_{12}^* = K_{21}^* = f_0(\eta_{\sigma,s})\cos(n_s, r_1). \tag{14}$$

This is a system of Fredholm integral equations of the second kind, which is solved and investigated in the usual way. All the calculations remain valid; the uniqueness theorem applies here just as in the ordinary Dirichlet problem.

Let us construct the integral equations for boundary conditions of the second kind. Suppose that the heat flux passing through the boundary of the plate is a prescribed function, namely $q_\sigma = F_1(\sigma)\cos\omega t + F_2(\sigma)\sin\omega t$. As before, we represent the solution of the problem posed in the form

$$T = \int_C \mu_1(s)K_3^*\,ds + \int_C \mu_2(s)K_4^*\,ds. \tag{15}$$

Set

$$K_3^* = f_0(\eta)\sin\omega t - g_0(\eta)\cos\omega t,$$
$$K_4^* = g_0(\eta)\sin\omega t + f_0(\eta)\cos\omega t. \tag{16}$$

Since $q(\sigma) = -\lambda h \dfrac{\partial T}{\partial n_\sigma}$, we have

$$\frac{\partial T}{\partial n_\sigma} = -2\mu_1(\sigma)\cos\omega t + 2\mu_2(\sigma)\sin\omega t + \int_C \mu_1(s)K_5^*(s,\sigma)\,ds +$$

$$+ \int_C \mu_2(s)K_6^*(s,\sigma)\,ds, \qquad (17)$$

where $K_5^* = dK_3^*/dn_\sigma$, $K_6^* = dK_4^*/dn_\sigma$ for $\eta = \eta_{\sigma,s}$. Hence we obtain two integral equations.

In an analogous manner one can obtain Fredholm equations for boundary conditions of the third kind.

2. Let us consider thermal waves in an infinite plate with boundary conditions of the third kind on the upper and lower surfaces. In this case, the heat-conduction equation in the absence of sources has the form

$$\lambda h \nabla^2 T = 2\delta T + c\rho h \frac{\partial T}{\partial t},$$

where δ is the heat-transfer coefficient. As before, we shall seek the solution in the form (2); moreover, after passing to dimensionless coordinates, φ satisfies the equation

$$\nabla^2\nabla^2\varphi - 2b_0\nabla^2\varphi + \varphi = 0, \qquad (18)$$

where

$$b_0 = \frac{2\delta\lambda h}{4\delta^2 + c^2\rho^2h^2\omega^2}, \quad \xi = \frac{r}{l_2}, \quad l_2 = \frac{\lambda h}{\sqrt{4\delta^2 + c^2\rho^2h^2\omega}}.$$

Here one may make use of the analogy with the problem of a plate on an elastic foundation, stretched by forces in the middle plane.

Under the action of a point source we obtain

$$\varphi(\xi,\theta) = \tilde{f}_0(\xi), \qquad (19)$$

where

$$\tilde{f}_n(\xi) + i\tilde{g}_n(\xi) = H_n^{(1)}(\xi e^{i\psi}), \quad \psi = \frac{1}{2}\,\mathrm{arctg}\,\frac{-b_0}{\sqrt{1-b_0^2}}.$$

In the case under consideration,

$$\varphi_1 = \frac{1}{\sin 2\psi}\left[\nabla^2\varphi + \varphi\cos 2\psi\right]$$

and, after simple transformations, we obtain

$$T = \frac{Q}{4\lambda h}\left[\tilde{f}_0(\xi)\cos\omega t - \tilde{g}_0(\xi)\sin\omega t\right].$$

Integrating this expression and carrying out the same calculations as those given above, we shall find the corresponding solutions; in doing so, many of the formulas given in ([1]) can be used almost without modification.

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CITED LITERATURE

[1] B. G. Korenev, *Questions of the Calculation of Beams and Plates on an Elastic Foundation*, Moscow, 1954.
[2] B. G. Korenev, DAN, 107, No. 2 (1956).

Note: Figure translations are in progress. See original paper for figures.