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Abstract

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THE THEORETICAL FOUNDATIONS FOR CALCULATING ICE DRIFT IN THE CENTRAL ARCTIC BASIN

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The true drift of ice in the Arctic basin is distinguished by exceptional complexity owing to the influence of a number of random factors, among which one may include, for example, pulsations of the wind and current, the shape, dimensions, and depth of immersion of ice floes, etc. It is therefore expedient to consider the ice drift, the wind, and the current averaged over a sufficiently long interval of time. We shall assume that, under such averaging (over a month or a season), the influence of random factors is eliminated. In addition, let us suppose that the indicated averaged state can be obtained by solving a stationary problem.

We shall write the equations of the established motion of seawater, ice, and air along horizontal Cartesian coordinate axes X, Y, neglecting nonlinear inertial terms and terms due to horizontal exchange of momentum, in the form

$$A\frac{\partial^2 u}{\partial z^2} + \Omega v = -g\frac{\partial \xi}{\partial x}; \qquad A\frac{\partial^2 v}{\partial z^2} - \Omega u = -g\frac{\partial \xi}{\partial y}; \tag{1}$$

$$T_x + R_x + \Omega\rho'' h v'' = 0; \qquad T_y + R_y - \Omega\rho'' h u'' = 0; \tag{2}$$

$$A'\frac{\partial^2 u'}{\partial z^2} + \Omega v' = \frac{1}{\rho'}\frac{\partial p}{\partial x}; \qquad A'\frac{\partial^2 v'}{\partial z^2} - \Omega u' = \frac{1}{\rho'}\frac{\partial p}{\partial y}, \tag{3}$$

where u, v are the horizontal components of velocity; p is atmospheric pressure; $z(x, y)$ is the deviation of the lower surface of the ice from its undisturbed horizontal position $z = 0$ (the Z-axis is directed downward); h is the thickness of the ice. In addition, g is the acceleration of gravity; Ω is the Coriolis parameter; ρ is density, and A is the coefficient of vertical exchange, taken to be constant. One and two primes indicate that the quantities under consideration refer, respectively, to air and to ice. Finally, R_x, R_y and T_x, T_y are the components of the tangential stress acting on the ice from the side of the water and the air. The latter are determined by the formulas

$$R_x = \rho A \left(\frac{\partial u}{\partial z}\right)_{z=\xi}; \qquad R_y = \rho A \left(\frac{\partial v}{\partial z}\right)_{z=\xi}; \tag{4}$$

$$T_x = -\rho' A' \left(\frac{\partial u'}{\partial z}\right)_{z=-h+\xi}; \qquad T_y = -\rho' A' \left(\frac{\partial v'}{\partial z}\right)_{z=-h+\xi}. \tag{5}$$

Boundary conditions. On the surface of the ice the no-slip conditions are satisfied; it is assumed that the current penetrates only to a depth $H = 200$ m, corresponding approximately to the position of the upper boundary of the Atlantic water mass; at greater heights the wind is geostrophic, and the horizontal gradients of atmospheric pressure are ···

are taken to be independent of height:

$$(u)_{z=\xi} = (u')_{z=-h+\xi} = u''; \qquad (v)_{z=\xi} = (v')_{z=-h+\xi} = v''; \tag{6}$$

$$(u)_{z=H} = (v)_{z=H} = 0; \tag{7}$$

$$(u')_{z\to-\infty} = -\frac{1}{\Omega\rho'}\frac{\partial p}{\partial y}; \qquad (v')_{z\to-\infty} = \frac{1}{\Omega\rho'}\frac{\partial p}{\partial x}. \tag{8}$$

The condition closing the system of equations (1)–(3), the mass-balance condition in the column from the upper surface of the ice to the lower boundary of the layer encompassed by the current,

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0, \tag{9}$$

in which

$$Q_x = \rho S_x + \rho'' h u'', \qquad Q_y = \rho S_y + \rho'' h v'', \tag{10}$$

where $S_x = \int_\xi^H u\,dz$ and $S_y = \int_\xi^H v\,dz$, makes it possible to introduce the function $\psi(x, y)$, defined by the formulas

$$Q_x = -\frac{\partial \psi}{\partial y}; \qquad Q_y = \frac{\partial \psi}{\partial x}. \tag{11}$$

The conditions at the boundary of the Central Arctic Basin L, which we shall set along the 200 m isobath, are formulated not for velocity, but only for the integral mass flux, since the boundary L is fluid. Where the shore is encountered along the normal n to L, we take $(Q_n)_L = 0$. Where the shore is not encountered, for example at the strait between Spitsbergen and Greenland or at Bering Strait, the value of $(Q_n)_L$, equal with quite sufficient accuracy to $\rho(S_n)_L$, is prescribed (the water discharges through the straits are known). We write the indicated conditions in the form

$$(\psi)_L = f(L), \tag{12}$$

where f is a known function on the contour L.

Since $H > 2D$, from (1), (4), and (7), with quite sufficient accuracy, for the layer from the lower surface of the ice to depth D we shall have (1):

$$u = \frac{e^{-a(z-\xi)}}{\rho a A\sqrt{2}}\left[-R_x \sin\alpha - R_y \cos\alpha\right] + \frac{g}{\Omega}\frac{\partial \xi}{\partial y};$$

$$v = \frac{e^{-a(z-\xi)}}{\rho a A\sqrt{2}}\left[R_x \cos\alpha - R_y \sin\alpha\right] - \frac{g}{\Omega}\frac{\partial \xi}{\partial x}, \tag{13}$$

where $D = \pi/a,\ a = \sqrt{\Omega/2A},\ \alpha = \pi/4 - a(z-\xi)$.

Further, from (1), (4), and (7) one may obtain the expressions

$$\frac{\partial \xi}{\partial x} = -\frac{1}{g\rho}\frac{2a}{1+(2aH-1)^2}\left[-(2aH-1)R_x - R_y - \Omega\rho S_x + \Omega\rho(2aH-1)S_y\right];$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{g\rho}\frac{2a}{1+(2aH-1)^2}\left[R_x - (2aH-1)R_y - \Omega\rho(2aH-1)S_x - \Omega\rho S_y\right], \tag{14}$$

which, taking into account (2), (10), and (11), we write in the form

$$\frac{\partial \xi}{\partial x} = -\frac{1}{g\rho}\frac{2a}{1+(2aH-1)^2}\left[(2aH-1)T_x + T_y + \Omega\frac{\partial \psi}{\partial y} + \Omega(2aH-1)\frac{\partial \psi}{\partial x}\right];$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{g\rho}\frac{2a}{1+(2aH-1)^2}\left[-T_x + (2aH-1)T_y + \Omega(2aH-1)\frac{\partial \psi}{\partial y} - \Omega\frac{\partial \psi}{\partial x}\right]. \tag{15}$$

Setting in (13) $z = \xi$, introducing $m = \frac{\rho''}{\rho}ah$, and taking into account (2), (6), and (15), we obtain the following expressions for the components of the ice-drift velocity:

$$u'' = \frac{T_x + (1+2m)T_y}{2\rho a A(1+2m+2m^2)} + \frac{2a}{\Omega\rho}\frac{1}{1+2m+2m^2}\frac{1}{1+(2aH-1)^2}\times$$
$$\times\left\{-(1+m)\left[-T_x-(2aH-1)T_y+\Omega(2aH-1)\frac{\partial\psi}{\partial y}-\Omega\frac{\partial\psi}{\partial x}\right]+\right. \quad (16)$$
$$\left.+m\left[(2aH-1)T_x+T_y+\Omega\frac{\partial\psi}{\partial y}+\Omega(2aH-1)\frac{\partial\psi}{\partial x}\right]\right\};$$

$$v'' = \frac{-(1+2m)T_x + T_y}{2\rho a A(1+2m+2m^2)} + \frac{2a}{\Omega\rho}\frac{1}{1+2m+2m^2}\frac{1}{1+(2aH-1)^2}\times$$
$$\times\left\{(1+m)\left[(2aH-1)T_x+T_y+\Omega\frac{\partial\psi}{\partial y}+\Omega(2aH-1)\frac{\partial\psi}{\partial x}\right]+\right.$$
$$\left.+m\left[-T_x+(2aH-1)T_y+\Omega(2aH-1)\frac{\partial\psi}{\partial x}-\Omega\frac{\partial\psi}{\partial y}\right]\right\}.$$

The function ψ entering into (16) is determined from the equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{2aH-1}{\Omega}\left(\frac{\partial T_y}{\partial x}+\frac{\partial T_x}{\partial y}\right)-\frac{1}{\Omega}\left(\frac{\partial T_x}{\partial x}+\frac{\partial T_y}{\partial y}\right), \quad (17)$$

obtained by eliminating ξ from (15) by cross differentiation.

Proceeding to the calculation of T_x and T_y, which also enter into (16), we write the solution of equations (3) under conditions (6) and (8) in complex form

$$w = w_g + (\mathfrak{w} - w_g)e^{s(z+h-\xi)}, \quad (18)$$

where $w = u' + iv'$; $w_g = \frac{i}{\Omega\rho'}\left(\frac{\partial p}{\partial x}+i\frac{\partial p}{\partial y}\right)$; $\mathfrak{w} = u'' + iv''$; $s^2 = \frac{i\Omega}{A'}$.

Since the velocity of the geostrophic wind is approximately 100 times greater than the ice-drift velocity, it follows from (18), with quite sufficient accuracy, that

$$w = w_g\left[1 - e^{s(z+h-\xi)}\right], \quad (19)$$

whence $(\partial w/\partial z)_{z=-h+\xi} = -sw_g$. Separating in the last expression the real and imaginary parts and taking (5) into account, we obtain

$$T_x = -\sqrt{\frac{A'}{2\Omega}}\left(\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}\right); \qquad T_y = \sqrt{\frac{A'}{2\Omega}}\left(\frac{\partial p}{\partial x}-\frac{\partial p}{\partial y}\right). \quad (20)$$

Substituting (20) into (17) and introducing an auxiliary function Φ, connected with the function ψ by the relation

$$\psi = \frac{H}{\Omega}\sqrt{\frac{A'}{A}}\,(p-\Phi), \tag{21}$$

we obtain Laplace' s equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0. \tag{22}$$

The boundary condition follows from (12) and (21):

$$(\Phi)_L=(p)_L-\frac{\Omega}{H}\sqrt{\frac{A'}{A}}\,f(L). \tag{23}$$

Substituting (20) into (16) and introducing the coefficients

$$K=\frac{1+m}{1+2m+2m^2}\,\frac{1}{\Omega\rho}\sqrt{\frac{A'}{A}};\qquad K'=\frac{1}{1/m+1}K; \tag{24}$$

$$\alpha=1-\frac{1}{2aH}+\frac{1}{2aH(2aH-1)},$$

we obtain

$$u''=-K\frac{\partial p}{\partial y}+K'\frac{\partial p}{\partial x}-\frac{K}{\alpha}\left\{(\alpha-1)\frac{\partial p}{\partial y}+\frac{1}{2aH-1}\left[\frac{\partial p}{\partial x}-\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial x}\right]\right.$$
$$\left.+\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial y}\right\}+\frac{K'}{\alpha}\left\{(\alpha-1)\frac{\partial p}{\partial x}-\frac{1}{2aH-1}\left[\frac{\partial p}{\partial y}-\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial y}\right]\right.$$
$$\left.+\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial x}\right\}; \tag{25}$$

$$v''=K\frac{\partial p}{\partial x}+K'\frac{\partial p}{\partial y}-\sqrt{\frac{K}{\alpha}}\left\{(\alpha-1)\frac{\partial p}{\partial x}-\frac{1}{2aH-1}\left[\frac{\partial p}{\partial y}-\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial y}\right]\right.$$
$$\left.+\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial x}\right\}+\frac{K'}{\alpha}\left\{(\alpha-1)\frac{\partial p}{\partial y}+\frac{1}{2aH-1}\left[\frac{\partial p}{\partial x}-\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial x}\right]\right.$$
$$\left.+\frac{\Omega}{H}\sqrt{\frac{A}{A'}}\,\frac{\partial\psi}{\partial y}\right\}.$$

For real values of the velocity of the averaged ice drift, $2aH > 100$; therefore expressions (25), with entirely sufficient accuracy, may be rewritten, taking (21) into account, in final form as

$$\begin{aligned} u'' &= -K\frac{\partial p}{\partial y} + K'\frac{\partial p}{\partial x} - K\frac{\partial(p-\Phi)}{\partial y} + K'\frac{\partial(p-\Phi)}{\partial x}; \\ v'' &= K\frac{\partial p}{\partial x} + K'\frac{\partial p}{\partial y} + K\frac{\partial(p-\Phi)}{\partial x} + K'\frac{\partial(p-\Phi)}{\partial y}. \end{aligned} \tag{26}$$

Thus, the determination of ice drift is reduced to integrating the Laplace equation (22) under the boundary condition (23) and to the subsequent computation of the drift-velocity components by formulas (26).

The first two terms on the right-hand side of (26) are the components of the velocity of purely wind-driven ice drift, i.e., drift caused directly by the dragging action of the wind on the ice. If the ice thickness is not taken into account ($m = 0$ and, consequently, $K' = 0$), then the purely wind-driven drift will be isobaric. In the opposite case, the purely wind-driven drift of the ice differs somewhat from isobaric drift (for example, for $m = 1/4$ we shall have $K' = 0.2K$). The subsequent terms are the components of the velocity of gradient ice drift, i.e., drift caused by the gradient current. Analysis of the drifts of the *Fram*, the *Sedov*, and the stations "North Pole 1–4" shows that, on average, the total ice drift consists by 2/3 of gradient drift and only by 1/3 of purely wind-driven drift. Hence it is clear how important the gradient drift of ice is, which was not taken into account in all previous theories devoted only to purely wind-driven drift (see, for example, [(2–4)]).

In conclusion, let us note that the generalization of the theory of total ice drift set forth here to the case of both variable coefficients of vertical exchange in the atmosphere and hydrosphere and variable penetration depth of the current presents no fundamental difficulties.

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CITED LITERATURE

[1] V. W. Ekman, *Ark. f. Mat., Astr. och Phys.*, **2**, No. 11 (1905); **17**, No. 26 (1923).
[2] V. V. Shuleikin, DAN, **19**, No. 8 (1938); **31**, No. 9 (1941).

[3] R. G. Gevorkyan, *Problems of the Arctic*, No. 4 (1941).
[4] M. E. Shvets, *Meteorol. and Hydrol.*, No. 6 (1946).

Note: Figure translations are in progress. See original paper for figures.