Mathematics

K. T. AKHMEDOV

1957

SovietRxiv

Abstract

Full Text

Mathematics

K. T. AKHMEDOV

ON THE CAUCHY PROBLEM FOR ONE CLASS OF NONLINEAR EQUATIONS IN FUNCTIONAL SPACES

(Presented by Academician S. L. Sobolev on 21 I 1957)

In the present work an equation of the form

$$\frac{du}{dt} = A\left(t, u, \frac{du}{dt}\right) + B(t, u), \tag{1}$$

is investigated, where u is an element of the Banach space X; du/dt also belongs to X; A is an operator analytic with respect to t, u and linear in du/dt for fixed values of t, u.

We shall seek a solution of equation (1) satisfying the condition

$$u\big|_{t=t_0} = u_0. \tag{2}$$

Making the substitution

$$u = u_0 + v, \qquad t = t_0 + \tau, \tag{3}$$

we obtain an equation in the new variables:

$$\frac{dv}{d\tau} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \tau^m A_{t_0,u_0}^{m,n}\left(v^n, \frac{dv}{d\tau}\right) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \tau^m B_{t_0,u_0}^{m,n}(v^n);$$

$$A_{t_0,u_0}^{m,n} = \frac{1}{m!\,n!}\,\frac{\partial^{m+n} A}{\partial t_0^m\,\partial u_0^n}; \qquad B_{t_0,u_0}^{m,n} = \frac{1}{m!\,n!}\,\frac{\partial^{m+n} B}{\partial t_0^m\,\partial u_0^n}, \tag{4}$$

where $A_{t_0,u_0}^{m,n}$ is an n-linear operator with respect to v and linear in $dv/d\tau$; $B_{t_0,u_0}^{m,n}$ is an n-linear operator with respect to v for each m.

Thus problem (1), (2) is reduced to the equivalent problem for equation (4) under the condition

$$v\big|_{\tau=0} = 0. \tag{2'}$$

Equation (4) may be represented as follows:

$$\begin{aligned}\frac{dv}{d\tau} = A^{0,0}_{t_0,u_0}\left(\frac{dv}{d\tau}\right) + \sum_{n=1}^{\infty} A^{0,n}_{t_0,u_0}\left(v^n, \frac{dv}{d\tau}\right) + \sum_{m=0}^{\infty} \tau^m A^{m,0}_{t_0,u_0}\left(\frac{dv}{d\tau}\right)\\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tau^m A^{m,n}_{t_0,u_0}\left(v^n, \frac{dv}{d\tau}\right) + B^{0,0}_{t_0,u_0} + \sum_{n=1}^{\infty} B^{0,n}_{t_0,u_0}(v^n)\\ + \sum_{m=1}^{\infty} B^{m,0}_{t_0,u_0}\tau^m + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tau^m B^{m,n}_{t_0,u_0}(v^n).\end{aligned} \tag{5}$$

For simplicity in the exposition of the results, let us consider a special case—a nonlinear integro-differential equation of the form

$$\frac{\partial u}{\partial t} = \int_0^1 K(t,x,s,u(s))\,\frac{\partial u}{\partial t}\,ds + f(t,x,u(x)). \tag{6}$$

Then equation (5) takes the form

$$\begin{aligned}\frac{\partial v}{\partial \tau} = \int_0^1 K^{0,0}_{t_0,u_0}(x,s)\frac{\partial v}{\partial \tau}\,ds + \int_0^1 \sum_{n=1}^{\infty} K^{0,n}_{t_0,u_0}(x,s)\frac{\partial v}{\partial \tau} v^n\,ds+\\ + \int_0^1 \sum_{m=1}^{\infty} K^{m,0}_{t_0,u_0}(x,s)\frac{\partial v}{\partial \tau}\tau^m\,ds + \int_0^1 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K^{m,n}_{t_0,u_0}(x,s)\tau^m v^n\,ds+\\ + f(t_0,x,u_0(x)) + \sum_{n=1}^{\infty} A^{0,n}_{t_0,u_0}(x)v^n + \sum_{m=1}^{\infty} A^{m,0}_{t_0,u_0}(x)\tau^m+\\ + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A^{m,n}_{t_0,u_0}(x)\tau^m v^n,\end{aligned} \tag{7}$$

where

$$K^{m,n}_{t_0,u_0}(x,s) = \frac{1}{m!\,n!}\,\frac{\partial^{m+n} K(t_0,x,s,u_0)}{\partial t_0^m\,\partial u_0^n};$$

$$A^{m,n}_{t_0,u_0}(x) = \frac{1}{m!\,n!}\,\frac{\partial^{m+n} f(t_0,x,u_0(x))}{\partial t_0^m\,\partial u_0^n}.$$

Theorem 1. If unity is not an eigenvalue of the kernel $K^{0,0}_{t_0,u_0}(x,s)$, then problem (6), (2) has a unique holomorphic solution $u(x,t)$ such that

$$\lim_{t \to t_0} u(x,t) = u_0(x).$$

Proof. We seek the solution of problem (7), (2′) in the form of a series

$$v(x,\tau) = \sum_{i=1}^{\infty} \tau^i v_i(x). \tag{8}$$

For the unique determination of the coefficients of the series (8), we obtain a recurrent system of equations of the form

$$\begin{aligned}
(n+1)v_{n+1} = (n+1)\int_0^1 K_{t_0,u_0}^{0,0}(x,s)v_{n+1}(s)\,ds + \\
+ \int_0^1 \sum_{\mu=1}^{n} \mu v_\mu(s) \left[\sum_{k=1}^{n-\mu+1} K_{t_0,u_0}^{0,k}(x,s) \left(\sum_{i=1}^{n-\mu+1} v_i(s) \right)^k \right]' ds + \\
+ \int_0^1 \sum_{h=1}^{n-1} \sum_{\mu=1}^{n-h} \mu v_\mu(s) \left[\sum_{k=1}^{n-h-\mu+1} K_{t_0,u_0}^{h,k}(x,s) \left(\sum_{i=1}^{n-h-\mu+1} v_i(s) \right)^k \right]' ds + \\
+ \int_0^1 \sum_{h=1}^{n} h K_{t_0,u_0}^{n-h+1,0}(x,s)v_h(s)\,ds + \sum_{\mu=1}^{n} \mu v_\mu(x) \left[\sum_{k=1}^{n-\mu+1} A_{t_0,u_0}^{0,k}(x) \left(\sum_{i=1}^{n-\mu+1} v_i(x) \right)^k \right]' + \\
+ \sum_{h=1}^{n-1} \sum_{\mu=1}^{n-h} \mu v_\mu(x) \left[\sum_{k=1}^{n-h-\mu+1} A_{t_0,u_0}^{h,k}(x) \left(\sum_{i=1}^{n-h-\mu+1} v_i(x) \right)^k \right]' + \\
+ \sum_{h=1}^{n} h A_{t_0,u_0}^{n-h+1,0}(x) v_h(x) * \quad \text{for } n \geqslant 1.
\end{aligned} \tag{9}$$

---

* $[\,]'$ means that, upon expanding the sums, one must retain those terms for which the sum of the products of the lower and upper indices of v is equal to the upper index of the summation sign Σ.

Assuming

$$\left|\Gamma_{t_0,u_0}(x,s)\right| \leqslant B_{t_0,u_0},$$

$$\left|\int_0^1 K^{m,n}_{t_0,u_0}(x,s)\,ds\right| \leqslant A,$$

$$\left|A^{m,n}_{t_0,u_0}(x)\right| \leqslant A \qquad (m,n=0,1,2,...)$$

and considering the majorant equation

$$\frac{d\psi}{d\tau} = A_1\left\{\frac{d\psi}{d\tau}\left[\frac{\psi}{1-\psi}+\frac{\tau}{1-\tau}+\frac{\psi\tau}{(1-\psi)(1-\tau)}\right]+\frac{\psi}{1-\psi}+\frac{\tau}{1-\tau}+\frac{\psi\tau}{(1-\psi)(1-\tau)}\right\}$$

under the condition $\psi|_{\tau=0} = 0$, we prove the convergence of the series (8); $\Gamma_{t_0,u_0}(x,s)$ is the resolvent of the kernel $K^{0,0}_{t_0,u_0}(x,s)$; $A_1 = A(1+B_{t_0,u_0})$ (cf. (5)).

Theorem 2. *If unity is an eigenvalue of the kernel $K^{0,0}_{t_0,u_0}(x,s)$ of rank 1 and if*

$$\int_0^1 A^{0,0}_{t_0,u_0}(x)\,q(x)\,dx = 0,$$

then the Cauchy problem (1), (2) has, generally speaking, two holomorphic solutions $u_1(x,t)$, $u_2(x,t)$ such that

$$\lim_{t\to t_0} u_1(x,t) = \lim_{t\to t_0} u_2(x,t) = u_0(x);$$

$q(x)$ is the eigenfunction of the kernel $K^{0,0}_{t_0,u_0}(x,s)$ corresponding to the eigenvalue unity.

Proof. Using the substitution (3), we obtain equation (7), and seek its solution in the form (8); we obtain the recurrent system of equations (9) for $n \geqslant 1$, while for $n = 0$ we have

$$v_1(x) = \int_0^1 K^{0,0}_{t_0,u_0}(x,s)v_1(s)\,ds + A^{0,0}_{t_0,u_0}(x).$$

Hence we find

$$v_1(x) = C_1 p(x) + A^{0,0}_{t_0,u_0}(x) + \int_0^1 H_{t_0,u_0}(x,s)A^{0,0}_{t_0,u_0}(s)\,ds.$$

Taking into account that C_1 is an arbitrary constant and that in the free term of the second equation $v_1(x)$ occurs squared, we find two values $C_{1,k}$ which ensure the orthogonality of the free term of the second equation and $p(x)$. Then

$$v_2(x) = C_2 p(x) + Q^{(1)}_{t_0,u_0,\alpha}(x),$$

where

$$Q^{(1)}_{t_0,u_0,\alpha}(x) = f_{1,\alpha}(x) + \int_0^1 H_{t_0,u_0}(x,s) f_{1,\alpha}(s)\, ds;$$

$f_{1,\alpha}(s)$ is the free term of the equation for $n \geqslant 1$.

If one takes into account that the solution immediately preceding enters into the free term of each subsequent equation to the first degree, then,

continuing the process, we find all the coefficients of the series (8) (generally speaking, ambiguously)

$$v_{n,\alpha}(x) = C_{n,\alpha} p(x) + Q^{(n-1)}_{t_0,u_0,\alpha}(x),$$

where

$$Q^{(n-1)}_{t_0,u_0,\alpha}(x) = f_{n-1,\alpha}(x) + \int_0^1 H_{t_0,u_0}(x,s) f_{n-1,\alpha}(s)\, ds;$$

$f_{n-1,\alpha}(s)$ is the free term of equation (9).

Next we prove the convergence of the series (8) with coefficients $v_{i,\alpha}(x)$ in some neighborhood of the point $\tau = 0$.

Let the homogeneous equation be given

$$\frac{\partial u}{\partial t} = \int_0^1 K(t,x,s,u(s)) \frac{\partial u}{\partial t}\, ds. \tag{10}$$

Theorem 3. *If:*

1) *equation (10) has the solution $u_0(x)$ for $t = t_0$;*
2) *unity is an eigenvalue of the kernel $K^{0,0}_{t_0,u_0}(x,s)$;*
3) $$\int_0^1 f(t_0,x,u_0(x)) q(x)\, dx \neq 0 \quad \text{and} \quad K^{0,1}_{t_0,u_0}(x,s) \neq 0,$$

then in a neighborhood of the point $t = t_0$ equation (6) has no solution holomorphic with respect to $t - t_0$. It has a solution

$$u_\alpha(x,t) = u_0(x) + \sum_{i=1}^{\infty}(t-t_0)^{i/2} v_{i,\alpha}(x),$$

which we shall call a special solution of the second order.

For the proof of the theorem we use the substitution

$$u = u_0 + v, \qquad t = t_0 + \tau^2.$$

One can prove a more general proposition:

If equation (10) has the solution $u_0(x)$ for $t = t_0$;

$$\int_0^1 f(t_0, x, u_0(x)) q(x)\, dx \neq 0; \qquad K^{0,\varkappa}_{t_0,u_0}(x,s) \equiv 0,$$

$$\varkappa = 1, 2, \dots, k-2; \qquad K^{0,k-1}_{t_0,u_0}(x,s) \neq 0,$$

then in a neighborhood of the point $t = t_0$ equation (6) has no solution holomorphic with respect to $t - t_0$. It has a solution

$$u_\alpha(x,t) = u_0(x) + \sum_{i=1}^{\infty}(t-t_0)^{i/k} v_{i,\alpha}(x),$$

where the number of branches is greater than two.

Moscow State University
named after M. V. Lomonosov

Received
11 I 1957

CITED LITERATURE

[1] N. N. Nazarov, *Tr. Sredneaziatsk. gos. univ.*, **5**, ser. matem., issue 33 (1941).
[2] N. N. Nazarov, *Tr. Inst. matem. AN UzSSR*, issue 4 (1948).
[3] K. T. Akhmedov, *Tr. Inst. fiz. i matem. AN AzerbSSR*, ser. matem., **7** (1955).
[4] K. T. Akhmedov, *Uch. zap. Azerb. gos. univ.*, No. 6 (1955).
[5] M. K. Gavurin, *Uch. zap. Leningr. gos. univ.*, ser. matem. nauk, issue 19, No. 137, 59 (1950).

Note: Figure translations are in progress. See original paper for figures.