



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

L. I. RUBINSTEIN

1957

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-195701.96589>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1957. Volume 117, No. 3

MATHEMATICAL PHYSICS

L. I. RUBINSTEIN

ON THE QUESTION OF THE UNIQUENESS OF THE SOLUTION OF THE ONE-DIMENSIONAL STEFAN PROBLEM IN THE CASE OF A ONE-PHASE INITIAL STATE OF A HEAT-CONDUCTING MEDIUM

(Presented by Academician S. L. Sobolev, 31 V 1957)

The following problem is considered:

$$\frac{\partial^2 U_1}{\partial x^2} = \frac{\partial U_1}{\partial t}; \quad 0 < x < y(t); \quad a^2 \frac{\partial^2 U_2}{\partial x^2} = \frac{\partial U_2}{\partial t}; \quad y(t) < x < 1; \quad (1_1)$$

$$U_1(0, t) = f_1(t) \leq 0; \quad U_2(x, 0) = \varphi(x) \geq 0; \quad U_2(1, t) = f_2(t) \geq 0; \quad (1_2)$$

$$U_1(y(t), t) = U_2(y(t), t) = 0; \quad (1_3)$$

$$\frac{dy}{dt} = \frac{\partial}{\partial x} U_1(y(t), t) - \frac{\partial}{\partial x} U_2(y(t), t); \quad y(0) = 0. \quad (1_4)$$

It is assumed that $f_i(t)$, $\varphi(x)$ are such that, for $0 \leq t \leq T$,

$$\frac{\partial}{\partial x} U_i \geq 0$$

($i = 1, 2$) everywhere in the domain of their existence and, moreover, that $|f_i(t)|$ are nondecreasing functions of t ($0 \leq t \leq T$).

Under these assumptions it was proved in ⁽¹⁾ that there do not exist two systems $U_{ij}(x, t); y_j(t)$ ($i, j = 1, 2$) of solutions of problem (1₁)–(1₄) for which $t = 0$ would not be a point of accumulation of zeros of the difference $y_1(t) - y_2(t)^*$.

Below, while retaining the assumptions made in ⁽¹⁾, we prove the uniqueness of the solution of problem (1₁)–(1₄).

§ 1. Let $\varepsilon > 0$ be arbitrarily small. Put

$$z_1(t) = y(t) - \varepsilon t/T; \quad z_2(t) = y(t) + \varepsilon t/T \quad (0 \leq t \leq T), \quad (2)$$

where $U_1(x, t), U_2(x, t), y(t)$ is some solution of the problem, and $T > 0$ is defined above. Next put:

$$\tau = t; \quad \xi = x - y(t) + z_1(t); \quad W_i(\xi, \tau) = U_i(x, t) \quad (i = 1, 2). \quad (3)$$

We find

$$\begin{aligned} \frac{\partial^2 W_1}{\partial \xi^2} &= \frac{\partial W_1}{\partial \tau} - \frac{\varepsilon}{T} \frac{\partial}{\partial \xi} W_1(\xi, \tau); & -\frac{\varepsilon \tau}{T} < \xi < z_1(\tau); \\ a^2 \frac{\partial^2 W_2}{\partial \xi^2} &= \frac{\partial W_2}{\partial \tau} - \frac{\varepsilon}{T} \frac{\partial}{\partial \xi} W_2(\xi, \tau); & z_1(\tau) < \xi < 1 - \frac{\varepsilon \tau}{T}; \end{aligned} \quad (4)$$

$$W_1|_{\xi=-\varepsilon\tau/T} = f_1(\tau); \quad W_2|_{\tau=0} = \varphi(\xi);$$

$$W_2|_{\xi=1-\varepsilon\tau/T} = f_2(\tau); \quad W_i|_{\xi=z_i(\tau)} = 0 \quad (i = 1, 2). \quad (4^*)$$

* The existence of a solution of problem (1₁)–(1₄) was proved under somewhat more general assumptions in ⁽²⁾. The restrictions that must be imposed on f_i and φ in order that the nonnegativity condition $\partial U_i / \partial x$ be satisfied are readily determined from the system of functional equations constructed in ⁽³⁾.

The problem (1₁)–(1₄) was also considered by A. Datsev ⁽⁴⁾, who, in proving the existence of a solution, allowed a number of substantial inaccuracies, and stated the uniqueness assertion without any justification.

By virtue of the assumption $\partial U_i / \partial x \equiv \partial W_i / \partial \xi \geq 0$ for $0 \leq t \leq T$. Consequently, $W_i(\xi, \tau)$ are superparabolic. Hence, and from the assumed monotonicity of $f_i(t)$, it follows, by the known maximum principle for subparabolic functions ⁽⁵⁾, that

$$W_1(0, \tau) \geq f_1(\tau). \quad (5)$$

Let now $V_{ij}(\xi, \tau)$ ($i = 1, 2$) be the solution of the problem obtained from (1₁)–(1₃) by replacing x by ξ , t by τ , and $y(t)$ by $z_j(\tau)$ ($i, j = 1, 2$). By virtue of the maximum principle, the parabolicity of $V_{21}(\xi, \tau)$, and the monotonicity of $f_2(\tau)$, it is evident that

$$V_{21}|_{\xi=1-\varepsilon\tau/T} \leq f_2(\tau). \quad (5^*)$$

Comparing $W_1(\xi, \tau)$ with $V_{11}(\xi, \tau)$ in the region $0 < \xi < z_1(\tau)$ and $W_2(\xi, \tau)$ with $V_{21}(\xi, \tau)$ in the region $z_1(\tau) < \xi < 1 - \varepsilon\tau/T$, we find, by virtue of the maximum principle, the boundary conditions (1₂) and (1₃), (5) and (5*), the superparabolicity of W_i , and the parabolicity of V_{ij} , that

$$0 \geq W_1(\xi, \tau) \geq V_{11}(\xi, \tau); \quad 0 \leq V_{21}(\xi, \tau) \leq W_2(\xi, \tau). \quad (6)$$

Hence, and from the fact that W_i and V_{21} vanish for $\xi = z_1(\tau)$, it follows that, for $\xi = z_1(\tau)$,

$$\frac{\partial W_1}{\partial \xi} \leq \frac{\partial V_{11}}{\partial \xi}; \quad \frac{\partial W_2}{\partial \xi} \geq \frac{\partial V_{21}}{\partial \xi}. \quad (7)$$

Now replacing, in the expression V_{i1} , ξ by x and τ by t , and taking into account the equalities $\partial W_i / \partial \xi = \partial U_i / \partial x$, we find, by virtue of (1₄), that

$$\frac{dz_1}{dt} = \frac{dy}{dt} - \frac{\varepsilon}{T} < \left[\frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial x} \right]_{x=y(t)} \leq \left[\frac{\partial V_{11}}{\partial x} - \frac{\partial V_{21}}{\partial x} \right]_{x=z_1(t)}. \quad (8)$$

In exactly the same way we show that

$$\frac{dz_2}{dt} = \frac{dy}{dt} + \frac{\varepsilon}{T} > \left[\frac{\partial V_{12}}{\partial x} - \frac{\partial V_{22}}{\partial x} \right]_{x=z_2(t)}. \quad (8^*)$$

Let now $U_1^*(x, t)$, $U_2^*(x, t)$, $y^*(t)$ be a second solution of problem (1₁)–(1₄). In (6) we proved that, if $f_1'(t)$ and $\varphi'(x)$ can be represented in the form

$$f_1'(t) = -c_0 t^{m_0} + c_1 t^{m_1} (1 + f^*(t)); \quad \varphi'(x) = b_0 x^{n_0} + b_1 x^{n_1} (1 + \varphi^*(x)),$$

where $\lim_{t \rightarrow 0} f^* = \lim_{x \rightarrow 0} \varphi^* = 0$; $c_0 > 0$; $b_0 > 0$; $m_0 > -1$; $n_0 > -1$; $m_1 > m_0$; $n_1 > n_0$, then for any solution of the problem

$$y'(t) = \gamma_0 t^{\nu_0} + \gamma_1 t^{\nu_1} (1 + y^*(t)),$$

where γ_0 and ν_0 are determined uniquely; $\lim_{t \rightarrow 0} y^* = 0$, $\nu_1 > \nu_0$. This assertion can be generalized. Namely, if

$$f_1'(t) = \Psi_1(t)t^{m_0}; \quad \varphi'(x) = \Psi_2(x)x^{n_0}, \quad (9)$$

where $m_0 > -1$, $n_0 > -1$, and for any $\eta > 0$

$$\lim_{t \rightarrow 0} \Psi_i(t)t^\eta = 0; \quad \lim_{t \rightarrow 0} |\Psi_i(t)|t^{-\eta} = \infty,$$

then

$$y'(t) = \gamma(t)t^{\nu_0}, \quad (10)$$

where γ_0 is determined uniquely, and

$$\lim_{t \rightarrow 0} \gamma(t)t^{n_1} = 0; \quad \lim_{t \rightarrow 0} \gamma(t)t^{-m} = \infty.$$

Moreover, $\gamma_0 > 0$, if $f_1(0) = 0$, which is assumed below.

Thus, without restricting generality by additional assumptions, one may assert that $y(t)$ and $y^*(t)$ are tangent at $t = 0$. Hence from (8) and (8*) it follows that there exists $t_0 > 0$, depending, generally speaking, on ε , such that

$$z_1(t) \leq y^*(t) \leq z_2(t) \quad \text{for } 0 \leq t \leq t_0. \quad (11)$$

Suppose now that the set \mathcal{E} of those t from the interval $(0, T)$ for which $y^*(t) < z_1(t)$ is nonempty. Let $T_1 = \inf \mathcal{E}$. Obviously, by continuity, $y^*(T_1) = z_1(T_1)$, and, however small $\delta > 0$ may be, in the interval $(T_1, T_1 + \delta)$ there is a t such that $z_1(t) > y^*(t)$. On the other hand, U_1^* is defined for $0 \leq t \leq T_1$ in the region $0 \leq x \leq z_1(t)$, and $V_{21}(x, t)$ in the region $y^*(t) \leq x \leq 1$. Comparing the boundary conditions determining U_i^* and V_{i1} , we find, by virtue of the maximum principle, that

$$0 \geq V_{11}(x, t) \geq U_1^*(x, t) \geq 0 \quad \text{for } 0 \leq x \leq z_1(t), \quad 0 \leq t \leq T_1;$$

$$0 \leq U_2^*(x, t) \leq V_{21}(x, t) \quad \text{for } y^*(t) \leq x \leq 1, \quad 0 \leq t \leq T_1.$$

Hence, and from the equalities

$$V_{11}(z_1(T_1), T_1) = U_1^*(z_1(T_1), T_1) = 0;$$

$$V_{21}(y^*(T_1), T_1) = U_2^*(y^*(T_1), T_1) = 0; \quad y^*(T_1) = z_1(T_1),$$

it follows that

$$\frac{\partial V_{11}}{\partial x} \Big|_{x=z_1(T_1)} \leq \frac{\partial U_1^*}{\partial x} \Big|_{x=y^*(T_1)} \quad \text{for } t = T_1;$$

$$\frac{\partial V_{21}}{\partial x} \Big|_{x=z_1(T_1)} \geq \frac{\partial U_2^*}{\partial x} \Big|_{x=y^*(T_1)} \quad \text{for } t = T_1.$$

But this, by virtue of (8) and (14), means that

$$\frac{dz_1}{dt} \Big|_{t=T_1} < \frac{dy^*}{dt} \Big|_{t=T_1}.$$

Hence, and from $y^*(T_1) = z_1(T_1)$, there follows the existence of a $\delta > 0$ such that for all $t \in (T_1, T_1 + \delta)$ $z_1(t) < y^*(t)$. But this contradicts the definition of T_1 . Consequently, for all $t \in (0, T)$,

$$z_1(t) \leq y^*(t). \quad (12)$$

In exactly the same way we prove that for all $t \in (0, T)$

$$z_2(t) \geq y^*(t). \quad (12^*)$$

But this means that for $0 \leq t \leq T$

$$|y(t) - y^*(t)| < \varepsilon t/T \leq \varepsilon,$$

whence, by virtue of the arbitrariness of $\varepsilon > 0$, it follows that $y(t) \equiv y^*(t)$, as was required to be proved.

§ 2. Let now, instead of the first of conditions (1₂), the condition

$$(\partial/\partial x - h)U_1(0, t) = -hf_1(t) \geq 0; \quad f_1(0) = 0. \quad (1'_2)$$

The existence of a solution of the problem obtained is established with the help of lemmas formulated in (7), analogously to how this was done in the case of problem (1₁)—(1₄) in (2)*.

The uniqueness of the solution is established analogously to the preceding one. Indeed, let $U_i(x, t), y(t)$ ($i = 1, 2$) be a solution of problem (1₂), (1₂*). Define again $z_i(t)$ by the equalities (2), and V_{ij} by the conditions

$$\frac{\partial^2 V_{1j}}{\partial x^2} = \frac{\partial V_{1j}}{\partial t}; \quad 0 < x < z_j(t); \quad a^2 \frac{\partial^2 V_{2j}}{\partial x^2} = \frac{\partial V_{2j}}{\partial t}; \quad z_j(t) < x < 1;$$

$$V_{ij}(0, t) = U_1(0, t); \quad V_{2j}(x, 0) = \varphi(x); \quad V_{2j}(1, t) = f_2(t); \quad (13)$$

$$V_{ij}(z_j(t), t) = 0 \quad (i, j = 1, 2).$$

It can be shown that, if $f_1(t)$ and $\varphi(x)$ satisfy the matching conditions, then $U_1(0, t)$ is monotone near $t = 0$. Using this and arguing in the same way as in § 1, we show that there exists a $T > 0$ such that, for $0 \leq t \leq T$,

$$\frac{dz_1}{dt} < \frac{\partial}{\partial x} V_{11} - \frac{\partial}{\partial x} V_{21} \Big|_{x=z_1(t)}; \quad \frac{dz_2}{dt} > \frac{\partial}{\partial x} V_{12} - \frac{\partial}{\partial x} V_{22} \Big|_{x=z_2(t)}. \quad (14)$$

At the same time, the inequalities

$$V_{11}(x, t) \leq U_1(x, t); \quad V_{12} \geq U_1(x, t)$$

hold everywhere in the region of their common definition. Using the fact that, by definition, $V_{ij}(0, t) = U_1(0, t)$, we find

$$\frac{\partial}{\partial x} V_{11} \Big|_{x=0} \leq \frac{\partial}{\partial x} U_1 \Big|_{x=0}; \quad \frac{\partial}{\partial x} V_{12} \Big|_{x=0} \geq \frac{\partial}{\partial x} U_1 \Big|_{x=0}.$$

But this means that

$$(\partial/\partial x - h)V_{11}|_{x=0} \leq -hf_1(t); \quad (\partial/\partial x - h)V_{12}|_{x=0} \geq -hf_1(t). \quad (15)$$

Comparing (2), (13), (14), and (15) with the conditions of lemma (2) from (7), and taking into account that any solution $y^*(t)$ different from $y(t)$ touches $y(t)$ at $t = 0$, we find that, for $0 \leq t \leq T$,

$$z_1(t) \leq y^*(t) \leq z_2(t). \quad (16)$$

But this again means that $y(t) \equiv y^*(t)$, which was required to be proved.

Let us note in conclusion that the restrictions under which we have proved the uniqueness theorems for solutions of problems (1₁)—(1₄) and (1_i)—(1₂^{*}) are very stringent and, apparently, are connected only with the method of proof adopted, and not with the essence of the matter. This is evident, for example, from the

fact that for the case of a two-phase initial state, when the uniqueness of the solution of the problem is established simultaneously with the convergence of Picard's iterative process ^(8,3), there is no need for similar restrictions. However, we do not yet have a proof of the uniqueness theorems free of these restrictions.

Ufa Petroleum Institute
Chernilovsk, Bashkir ASSR

Received
17 IX 1956

REFERENCES

1. L. I. Rubinshtein, DAN, 79, No. 1 (1951).
2. L. I. Rubinshtein, DAN, 62, No. 2 (1948).
3. L. I. Rubinshtein, DAN, 58, No. 2 (1947).
4. A. Datsev, DAN, 87, No. 3 (1952).
5. J. Petrowsky, *Compositio Math.*, 1, 383 (1935).
6. L. I. Rubinshtein, DAN, 62, No. 6 (1948).
7. L. I. Rubinshtein, DAN, 72, No. 1 (1951).
8. L. I. Rubinshtein, *Izv. AN SSSR, ser. geogr. i geofiz.*, 11, issue 1 (1947).

* The existence of a solution is guaranteed under the condition

$$\varphi(0) = 0; \quad \varphi'(x) = b_0 x^{n_0} (1 + \varphi^*(x)); \quad n_0 \geq 0; \quad \lim_{x \rightarrow 0} \varphi^* = 0;$$

$$f'(t) = -c_0 t^{m_0} (1 + f_1^*(t)); \quad m_0 \geq 0; \quad \lim_{t \rightarrow 0} f_1^* = 0; \quad m_0 \leq \frac{n_0}{2};$$

$$hc_0 > (2a)^{n_0} \Gamma\left(\frac{n_0 + 1}{2}\right) \frac{1}{\sqrt{\pi}} \quad \text{when } m_0 = \frac{n_0}{2}.$$

If these conditions are not satisfied, then for some time the medium remains one-phase.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.