AN EXAMPLE OF A SEQUENCE OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract

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MATHEMATICS

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AN EXAMPLE OF A SEQUENCE OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS

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Consider a sequence of functions

$$\varphi_n(y), \qquad n = 1, 2, \ldots,$$

each of which has the following properties:

a) it is analytic in the closed disk of radius $R > 0$ with center at the point $M(R; 0)$;

b) $\varphi_n(0) = 1$;

c) $(-1)^k \varphi_n^{(k)}(x) \geq 0,\ k = 0, 1, 2, \ldots,\ x \in [0, R]$;

d)

$$-\varphi_n^{(k)}(x) = n\varphi_n^{(k-1)}(x)\,(1 + \alpha_{kn}(x)), \qquad k = 1, 2, \ldots,$$

where $x \in [0, R]$ and $\alpha_{kn}(x)$ tends to zero uniformly with respect to k and x as n tends to infinity;

e)

$$\lim_{n\to\infty} \frac{n}{n_n} = 1.$$

Expand the function $\varphi_n(y)$ in a Taylor series in powers of $(y - x)$ $(x \in [0, R])$:

$$\varphi_n(y) = \sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(x)}{k!}(y - x)^k.$$

From conditions a) and b) it follows that

$$1 = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k. \qquad (1)$$

Consider the sequence of linear operators

$$L_n(f;x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k f\left(\frac{k}{n}\right), \qquad n = 1, 2, \dots, \qquad (2)$$

which will be positive on the basis of condition c).

We shall show that the sequence (2) converges uniformly on the interval $[0, R]$ for each of the three functions $1, x$, and x^2.

Uniform convergence to the function $f(x) = 1$ follows from identity (1). For the function $f(x) = x$ we have

$$L_n(y;x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi_n^{(k)}(x)}{k!} x^k \frac{k}{n}.$$

On the basis of condition d),

$$L_n(y;x) = x \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_n^{(k-1)}(x)}{(k-1)!} x^{k-1} (1 - \alpha_{kn}(x)) =$$

$$= x \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_n^{(k-1)}(x)}{(k-1)!} x^{k-1} + x \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_n^{(k-1)}(x)}{(k-1)!} x^{k-1} \alpha_{kn}(x).$$

Since

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{\varphi_{m_n}^{(k-1)}(x)}{(k-1)!} x^{k-1} = \varphi_{m_n}(0) = 1, \qquad (3)$$

and the sum

$$\sum_{k=1}^{\infty} (-1)^k \frac{\varphi_{m_n}^{(k-1)}(x)}{(k-1)!} x^{k-1} \alpha_{kn}(x),$$

by virtue of the conditions imposed on $\alpha_{kn}(x)$, tends uniformly to zero, it follows that $L_n(y;x)$ converges uniformly to x.

For the function $f(x) = x^2$ we have

$$L_n(y^2;x) = \sum_{k=0}^{\infty}(-1)^k \frac{\varphi_n^{(k)}(x)}{k!}\, x^k \left(\frac{k}{n}\right)^2$$
$$= x\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{m_n}^{(k-1)}(x)}{(k-1)!}\, x^{k-1}(1+\alpha_{kn}(x))\frac{k}{n}$$
$$= x^2\frac{m_n}{n}\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{m_{nm}n}^{(k-2)}(x)}{(k-2)!}\, x^{k-2}(1+\alpha_{kn}(x))(1+\alpha_{k-1\,m_n}(x))$$
$$+\frac{x}{n}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{m_n}^{(k-1)}(x)}{(k-1)!}\, x^{k-1}(1+\alpha_{kn}(x)).$$

The uniform convergence of $L_n(y^2;x)$ to x^2 follows from condition d), (3), from the fact that

$$\sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{m_{nm}n}^{(k-2)}(x)}{(k-2)!}\, x^{k-2} = \varphi_{m_{nm}n}(0) = 1,$$

and the sums

$$1)\quad \sum_{k=2}^{\infty}(-1)^{k-2}\frac{\varphi_{m_{nm}n}^{(k-2)}(x)}{(k-2)!}\, x^{k-2}(\alpha_{kn}(x)+\alpha_{k-1\,m_n}(x)+\alpha_{kn}(x)\alpha_{k-1\,m_n}(x)),$$

$$2)\quad \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\varphi_{m_n}^{(k-1)}(x)}{(k-1)!}\, x^{k-1}\alpha_{kn}(x)$$

converge uniformly to zero.

On the basis of P. P. Korovkin' s theorem ([1]), the sequence of operators (2) converges uniformly on the interval $[0,R]$ to the function $f(x)$, continuous on this interval, continuous at the point R from the right, and bounded on the set $x \geq 0$.

Let us consider some special cases.

A. Taking $\varphi_n(y) = (1-y)^n$, we obtain the Bernstein polynomials

$$B_n(f;x) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) C_n^k x^k (1-x)^{n-k} \tag{4}$$

and his theorem.

B. Taking $\varphi_n(y) = e^{-ny}$, we obtain the sequence of operators

$$L_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{n^k}{k!} x^k, \tag{5}$$

which converges uniformly on the interval $[0, R]$ $(R > 0)$ to the function $f(x)$, continuous on $[0, R]$, continuous from the right at the point R, and bounded on $[0, \infty)$. These operators have been considered by many authors, for example G. M. Mirakyan ([2]), Favard ([4]), Szasz ([4]), Băcer ([6]), and others.

c. Taking $\varphi_n = \dfrac{1}{(1+y)^n}$, we obtain the sequence of operators

$$L_n(f;y) = \frac{1}{(1+x)^n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{n(n+1)\ldots(n+k-1)}{k!} \left(\frac{x}{1+x}\right)^k, \tag{6}$$

converging uniformly on the interval $[0, R]$ to the function $f(x)$, continuous on $[0, R]$, continuous at the point R from the right, and bounded on $[0, \infty)$.

The following theorems describe the character of convergence of the operators (2).

Theorem 1. *If the following conditions are satisfied: 1) $f(x)$ is continuous on $[0, R]$; 2) $f(x) = f(R)$, if $x > R$; 3) $\alpha_{kn}(x) = \alpha_n$, then*

$$|L_n(f;x) - f(x)| \leq$$

$$\leq \omega\left(\frac{1}{\sqrt{n}}\right) \left[\sqrt{n} \sqrt{x^2 \frac{m_n}{n}(1+\alpha_n)(1+\alpha_{m_n}) + \frac{x}{n}(1+\alpha_n) - 2x^2(1+\alpha_n) + x^2 + 1}\right],$$

where $\omega(\delta)$ is the modulus of continuity of $f(x)$.

Theorem 2. *If the following conditions are satisfied: 1) the function $f(x)$ is continuous on $[0, R]$, continuous at the point R from the right, bounded on the set $x > 0$, and has a finite second derivative at the point $x \in [0, R]$; 2) $m_n = n + c$, where c is an integer; 3) $\alpha_{kn}(x) \equiv 0$, then*

$$L_n(f;x) = f(x) + \frac{f''(x)}{2!} \frac{x + cx^2}{n} + o\left(\frac{1}{n}\right).$$

Corollary 1. For the polynomials of S. N. Bernstein (4), for which $c = -1$, we have

$$B_n(f;x) = f(x) + \frac{f''(x)}{2!} \frac{x(1-x)}{n} + o\left(\frac{1}{n}\right).$$

This proves the theorem of E. V. Voronovskaya ([3]).

Corollary 2. For the operators (5), $c = 0$ and, consequently,

$$L_n(f;x) = f(x) + \frac{f''(x)}{2!}\frac{x}{n} + o\left(\frac{1}{n}\right).$$

Corollary 3. For the operators (6), $c = 1$ and, consequently,

$$L_n(f;x) = f(x) + \frac{f''(x)}{2!}\frac{x(x+1)}{n} + o\left(\frac{1}{n}\right).$$

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Note: Figure translations are in progress. See original paper for figures.