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Abstract

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ON PERIODIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH TIME DELAY

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Periodic solutions of quasilinear differential equations with delayed argument t have been considered in a number of works ([1−5]). In computing periodic solutions, an important role is played by theorems on the existence and uniqueness of such solutions corresponding to a given generating oscillation ([1]). In the case of a complex generating system, theorems on the existence and uniqueness (or isolation) of the generating oscillation are important. The purpose of the present note is to show the possibility of solving such problems for equations with time delays by the method of Lyapunov functions, similarly to how this is done for ordinary equations ([6]). Here only the rough case of asymptotic stability is considered, but for systems of a fairly general form. Sufficient conditions are given for the existence and uniqueness of a periodic solution for the generating system. It is shown that under these conditions there exists a unique periodic solution of the full system, tending to the generating one as $\mu \to 0$ *. This makes it possible, in the case considered, to justify the method of approximate computation of a periodic solution.

Consider the system of equations

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} p_{ij}(t) x_j(t - h_{ij}(t)) +$$

$$+\varphi_i(x_1(t - h_{i1}(t)), \dots, x_n(t - h_{in}(t)), t) + f_i(t) \tag{1}$$

$$\varphi_i(0, \dots, 0, t) = 0 \qquad (i = 1, 2, \dots, n),$$

where $p_{ij}(t)$, $\varphi_i(x_1, \dots, x_n, t)$, $f_i(t)$ are continuous, and $h_{ij}(t)$ are piecewise-continuous periodic functions of time with period T; $0 \le h_{ij}(t) \le H$. Obviously, equations (1) also include systems of equations in finite differences.

Together with system (1), consider the equations

$$\frac{dy_i}{dt} = \sum_{j=1}^{n} q_{ij}(t) y_j(t - g_{ij}(t))$$

$$(i = 1, 2, \dots, n), \tag{2}$$

where $q_{ij}(t)$ are continuous, and $g_{ij}(t) \geq 0$ are piecewise-continuous periodic functions.

* The results presented in this note overlap with the results of L. E. Elsgolts, obtained by another method (Periodic solutions of quasilinear differential equations with delayed argument. Report at the Third All-Union Mathematical Congress).

Theorem 1. If the solution $y_1 = \dots = y_n = 0$ of equations (2) is asymptotically stable*, then one can indicate positive numbers δ, γ, and L such that, when the inequalities

$$|p_{ij}(t) - q_{ij}(t)| < \delta, \qquad |h_{ij}(t) - g_{ij}(t)| < \gamma,$$

$$|\varphi_i(x''_1, \dots, x''_n, t) - \varphi_i(x'_1, \dots, x'_n, t)| < L \sum_{j=1}^{n} |x''_j - x'_j| \tag{3}$$

are satisfied, the system of equations (1) has a unique periodic solution, asymptotically stable in the sense of Lyapunov.

Remark. In the case where there are no delays ($h_{ij} = g_{ij} = 0$), the theorem coincides with known results (8).

Proof. Let $h = \sup[h_{ij}(t), g_{ij}(t)]$. We shall consider the trajectories of (1) and (2) in the space $\{x(\tau)\}$ (respectively $\{y(\tau)\}$), where $x(\tau)$ $(y(\tau))$ is an n-dimensional vector whose components are continuous functions $x_i(\tau)$ $(y_i(\tau))$ for $-2h \leqslant \tau \leqslant 0$; the metric is defined by the norm

$$\|x(\tau)\| = \sup[|x_i(\tau)|] \quad (-2h \leqslant \tau \leqslant 0).$$

The element of a trajectory corresponding to the moment t will be the curve $x(t+\tau)$ (respectively $y(t+\tau)$) for $-2h \leqslant \tau \leqslant 0$. For the questions considered here it is immaterial from what moment of time the consideration of solutions begins; therefore one may always assume the solution to be defined for $t_0 - 2h \leqslant t \leqslant t_0$. Everywhere below the functions of the variable τ are defined for $-2h \leqslant \tau \leqslant 0$.

Let $\theta = \max[T, 4h]$. The elements of the trajectories (2), $y(t_0+\theta+\tau)$, with initial data $\|y(t_0+\tau)\| = 1$, are contained in a family of functions $\{z(\tau)\}$ satisfying the conditions

$$\|z(\tau)\| \leqslant M, \qquad |z_i(\tau_2) - z_i(\tau_1)| \leqslant M|\tau_2 - \tau_1|,$$

where M is a constant. Using the compactness of $\{z(\tau)\}$ and of the interval $0 \leqslant t_0 \leqslant T$, the continuous dependence of the solutions of (2) on the initial data (for $t \geqslant t_0$) (9), and the linearity of equations (2), it is not difficult to verify the validity of the inequalities

$$\|y(t+\tau)\| < B\|y(t_0+\tau)\|e^{-\alpha(t-t_0)} \quad \text{for } t \geqslant t_0,\ 0 \leqslant t_0 \leqslant T \tag{4}$$

($\alpha > 0$, B are constants). Under these conditions there exists a functional $v(x(\tau), t)$ satisfying the estimates (10)

$$\lim_{\Delta t \to +0} \frac{\Delta v(y(t+\tau), t)}{\Delta t} < -\frac{1}{2}\|y(t+\tau)\|; \tag{5}$$

$$|v(x''(\tau), t) - v(x'(\tau), t)| < N\|x''(\tau) - x'(\tau)\|;$$

$$A\|x(\tau)\| < v(x(\tau), t) < K\|x(\tau)\|, \tag{6}$$

where N, K, A are positive constants.

Substituting into the functional v, instead of $x(\tau)$, the elements of the solution $x(t+\tau)$ of system (1), we obtain the estimate

$$\lim_{\Delta t \to +0} \frac{\Delta v}{\Delta t}\Big|_{(1)} = \lim_{\Delta t \to +0} \frac{\Delta v}{\Delta t}\Big|_{(2)} + (N_1\delta + N_2\gamma + N_3 L)\|x(t+\tau)\| + N_4 F$$

$$\text{for } t > t_0 + \theta, \tag{7}$$

where $F = \max \|f_i(t)\|$; N_1, N_2, N_3, N_4 are constants.

An analogous estimate is obtained if, instead of $x(\tau)$, one substitutes the difference $x''(t+\tau) - x'(t+\tau)$ of two solutions of (1), namely

$$\lim_{\Delta t \to +0} \frac{\Delta v}{\Delta t}\Big|_{\text{along } x''(t) - x'(t)} \leqslant \left(-\frac{1}{2} + N_1\delta + N_2\gamma + N_3 L\right)\|x''(t+\tau) - x'(t+\tau)\|$$

$$\text{for } t \geqslant t_0 + \theta. \tag{8}$$

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* What is meant is the definition of stability from article (7).

From estimates (6), (7), (8) there follows the existence of numbers $C > 0$ and m (m is a natural number) such that the mapping $x(t_0 + \tau) \to x(t_0 + mT + \tau)$ of the family of functions $v(x(t_0 + \tau), t_0) \leq C$ is a contraction [11], if the numbers δ, γ, L satisfy the inequality $\delta N_1 + \gamma N_2 + L N_3 < 1/2$. Thus there exists one and only one initial curve $x^*(t_0 + \tau) \to x^*(t_0 + mT + \tau) = x^*(t_0 + \tau)$, i.e., there exists a unique (by virtue of the contraction of the mapping–asymptotically stable) periodic solution $x^*(t + \tau)$ with period lT ($l \leq m$). Using the uniqueness of the periodic solution, it is not difficult to verify that the number l can be chosen equal to 1. The theorem is proved.

From the estimates given above there follows the uniform boundedness of the periodic solutions corresponding to different functions $f_i(t)$, provided only that these functions are bounded in modulus by one and the same number F.

We shall call a periodic solution $x^*(t, \mu)$ of the system of equations

$$\frac{dx_i}{dt} = X_i(x_1(t - h_{i1}(t, \mu)), \dots, x_n(t - h_{in}(t, \mu)), t, \mu) \tag{9}$$

stable with respect to the parameter μ, if for $\mu = 0$ this system has an isolated periodic solution $x^*(t, 0)$ and if for any $\varepsilon > 0$ one can indicate a number $\delta > 0$ such that, for all $|\mu| < \delta$, system (9) has a unique periodic solution such that $|x_i^*(t, 0) - x_i^*(t, \mu)| < \varepsilon$. In the absence of delays, in the general case, a periodic solution is known [12] to be unstable with respect to μ. However, in the particular case considered here such stability does occur.

Theorem 2. *If the solution $y_1 = \dots = y_n = 0$ of the system of equations (2) is asymptotically stable, then the periodic solution of the equations*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} (q_{ij}(t) + \mu p_{ij}(t)) x_j(t - [g_{ij}(t) + \mu h_{ij}(t)]) + f_i(t) +$$

$$+ \mu \varphi_i(x_1(t - [g_{ij}(t) + \mu h_{ij}(t)]), \dots, x_n(t - [g_{ij}(t) + \mu h_{ij}(t)]), t) \tag{10}$$

is stable with respect to the parameter μ.

Here $p_{ij}, q_{ij}, f_i, \varphi_i$ are continuous, while h_{ij}, g_{ij} are piecewise-continuous periodic (of period T) bounded functions of time; moreover, the functions φ_i satisfy conditions (3). Since only equations with delayed argument are considered, it is assumed that the inequality

$$g_{ij}(t) + \mu h_{ij}(t) \geq 0$$

is satisfied.

Proof. The existence and uniqueness of a periodic solution $x^*(t,\mu)$ of system (10) for $|\mu| < \mu_0$ (μ_0 is some positive constant) follows from Theorem 1. The difference $x(t,\mu) - x(t,0)$ satisfies the equations ($x(t,\mu)$ is any solution of (10))

$$\frac{d[x_i(t,\mu) - x_i(t,0)]}{dt} = \sum_{j=1}^{n} q_{ij}(t)(x_j(t - g_{ij}(t),\mu) - x_j(t - g_{ij}(t),0)) + R_i,$$

where the additional terms R_i satisfy the inequality (for $t > t_0+\theta$) $|R_i| < \mu R$ (R is a constant) for all $x(t,\mu)$ satisfying the condition $\|x(t+\tau,\mu) - x^*(t+\tau,0)\| < \eta$, where η is a sufficiently small positive number. Substituting into the functional v (5), in place of $x(\tau)$, the difference $x(t+\tau,\mu) - x^*((t+\tau),0)$, we obtain the estimate

$$\lim_{\Delta t \to +0} \frac{\Delta v}{\Delta t} < -\frac{1}{2}\|x(t+\tau,\mu) - x^*(t+\tau,0)\| + \mu N_5 R$$

(N_5 is a constant), from which it follows that, for sufficiently small μ, the trajectory $x(t,\mu)$ will not leave the ε-neighborhood of the solution $x^*(t,0)$, provided only that the initial curves $x^*(t_0+\tau,0)$ and $x(t_0+\tau,\mu)$ are sufficiently close. Since $x(t,\mu) \to x^*(t,\mu)$ as $t \to \infty$ (by Theorem 1), the periodic solution $x^*(t,\mu)$ must also lie in the ε-neighborhood of $x^*(t,0)$. The theorem is proved.*

Suppose now that the functions $h_{ij}, g_{ij}, f_i, \varphi_i, q_{ij}, p_{ij}$ have continuous derivatives up to order $n+1$, inclusive. Sometimes it is expedient to seek the periodic solution $x^*(t,\mu)$ of system (10) in the form of the sum

$$x^*(t,\mu) = x_{(0)}(t) + \mu x_{(1)}(t) + ... + \mu^n x_{(n)}(t) + R_n(t,\mu). \qquad (11)$$

The sum $\sum_{j=1}^{n} x_{(j)}(t)\mu^j$ represents the solution $x^*(t,\mu)$ to accuracy up to μ^n in the sense that $R_n(t,\mu) = o(\mu^n)$. Indeed, substituting (11) into (10), expanding $x_{(k)}(t - [g + \mu h])$ in powers of μ, and equating coefficients of like powers of μ, we obtain equations for $x_{(k)}(t)$ ($k = 1, 2, \dots, n$) and $x_{n+1}(t) = R_n(t,\mu)/\mu^{n+1}$:

$$\frac{dx_{i(k)}}{dt} = \sum_{j=1}^{n} q_{ij}(t) x_{j(k)}(t - g_{ij}(t)) + f_{i(k)}(t) \qquad (k = 0, 1, 2, \dots, n):$$

$$(12)$$

$$\frac{dx_{i(n+1)}}{dt} = \sum_{j=1}^{n} q_{ij}(t) x_{j(n+1)}(t - [g_{ij}(t) + \mu h_{ij}(t)]) + f_{i(n+1)}(t) +$$

$$+\mu\psi_i(x_{j(n+1)}(t-[g_{ij}(t)+\mu h_{ij}(t)]),(t,\mu)),$$

where the periodic functions $f_{i(k)}, \psi_i$ are expressed in terms of $h_{ij}, f_i, x_{i(l)}(t), x_{j(l)}^{(m)}, \partial^m\varphi_i/\partial x_{j_1}^{m_1} \ldots \partial x_{j_r}^{m_r}$ for $l<k,\ m\leq k$. According to Theorem 1, each equation (12), for $|\mu|<\mu_0$, has a unique periodic solution $x_{(k)}(t)$, $x_{(n+1)}(t,\mu)$; moreover, by the remark at the end of the proof of Theorem 1, the solution $x_{(n+1)}(t,\mu)$ is bounded uniformly with respect to μ for $|\mu|<\mu_0$, which proves our assertion. The representation of the solution in the form of the sum $\sum_{j=1}^{n} x_{(j)}(t)\mu^j$ is easily computed in the case where q_{ij} and g_{ij} are constants, and $p_{ij}, h_{ij}, \varphi_i, f_i$ are trigonometric polynomials, since, evidently, in this case the periodic solutions of the first n systems (12) have the form $x_{i(k)}=\sum a_\nu \exp\left(i\frac{2\pi}{T}\nu t\right)$.

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* Here the existence of a **unique** periodic solution of system (10) has been proved. If conditions (3) are satisfied only in a neighborhood of the generating solution $x(t, 0)$, then uniqueness of the periodic solution $x(t, \mu)$ will be proved only in a neighborhood of this generating solution.

Note: Figure translations are in progress. See original paper for figures.