MATHEMATICS

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1957

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Abstract

Full Text

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ON MATRIX SCHEMES

(Presented by Academician M. V. Keldysh on 6 X 1956)

We consider finite sets of objects $A_1, \dots, A_n$, which we shall call operators, and of two-valued (logical) variables $p_1, \dots, p_k$, taking the values 0 and 1.

Let $\Delta_1, \dots, \Delta_{2^k}$ denote all possible sets of values of the variables $p_1, \dots, p_k$. We shall say that an order of execution of the operators $A_1, \dots, A_n$ depending on the values of the logical variables $p_1, \dots, p_k$ is given if $n+1$ functions of the form $N_i(s) = j$ are given, where $i = 0, 1, \dots, n$; $s = 1, 2, \dots, 2^k$; $1 \leq j \leq n+1$. Thus, for every sequence of sets

$$\Delta_{s_1}, \Delta_{s_2}, \dots, \Delta_{s_m}, \Delta_{s_{m+1}}, \dots \tag{1}$$

there is defined a sequence of operators (finite or infinite):

$$A_{i_1}, A_{i_2}, \dots, A_{i_m}, A_{i_{m+1}}, \dots, \tag{2}$$

where $i_{l+1} = N_{i_l}(s_{l+1})$ $(l = 0, 1, 2, \dots;\ i_0 = 0)$; moreover, if for some l, $i_l = N_{i_{l-1}}(s_l) = n+1$, then the sequence (2) terminates at the $(l-1)$-st term.

Any order of execution of the operators $A_1, \dots, A_n$ depending on the values of the variables $p_1, \dots, p_k$ can be written in the form of a matrix

$$\begin{array}{c|cccc} & A_1 & A_2 & \cdots & A_n \\ \hline A_0 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n} \\ A_1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{array} \tag{3}$$

where $\alpha_{ij} = \alpha_{ij}(p_1, \dots, p_k)$ $(i = 0, 1, \dots, n;\ j = 1, \dots, n)$ are certain logical functions of the variables $p_1, \dots, p_k$, and we shall agree to understand A_0 as an empty operator symbolizing the beginning of the process. Matrices of the form (3), as well as those obtained from them by deleting or permuting certain rows or columns, will be denoted briefly by

$$A_i \,\Big|\, \alpha_{ij}.$$

The present work is devoted to the study of matrices specifying an order of execution of operators, and to their application in the theory of program schemes (p.s.).

Definition 1. Consider an arbitrary matrix $\mathfrak{A}(p_1, \dots, p_k) \equiv$

$$A_i \;\Big|\; \alpha_{ij}.$$

The **value** of the matrix $\mathfrak{A}$ for the sequence of sets (1) will mean the row of operators obtained as a result of the following process. Step 1: we examine the values of the elements α_{0j} $(j = 1, \dots, n)$ of the 0-th row* for the assignment Δ_{s_1}, and write down one of those operators, for example A_{i_1}, for which $\alpha_{0i_1}(\Delta_{s_1}) = 1$. Suppose m steps have been carried out, and at the m-th step the operator A_{i_m} was written down. Then at the $(m+1)$-st step we examine the values of the elements of the i_m-th row under the assignment $\Delta_{s_{m+1}}$ and write down the operator $A_{i_{m+1}}$ for which $\alpha_{i_m i_{m+1}}(\Delta_{s_{m+1}}) = 1$. The process is terminated if all elements of the row being examined under the given assignment have value 0, or if there is no row with the required number in the matrix.

Definition 2. A sequence of assignments (1) will be called **admissible** for the matrix $\mathfrak{A}(p_1, \dots, p_k)$ under the distribution of shifts

$$A_i - \mathfrak{P}_i \subset \{p_1, \dots, p_k\}^*, \tag{4}$$

if for this sequence there is a value $A_{i_1}, \dots, A_{i_m}, A_{i_{m+1}}, \dots$ (2) of the matrix $\mathfrak{A}$ such that, for every $m = 1, 2, \dots$, the assignment $\Delta_{s_{m+1}}$ differs from the assignment Δ_{s_m} in the values of variables only from $\mathfrak{P}_{i_m}$. Moreover, if the value (2) is finite and A_{i_r} is its last operator, then the assignments $\Delta_{s_{r+2}}, \Delta_{s_{r+3}}, \dots$ are arbitrary.

Definition 3. The matrix $\mathfrak{A}(p_1, \dots, p_k)$ under the distribution of shifts (4) will be called a **matrix scheme** (m.s.) if, for every admissible sequence, it has only one value (which may be empty).

An effective definition of an m.s. can be obtained by means of the following apparatus. Consider, for the matrix $\mathfrak{A}(p_1, \dots, p_k, A_1, \dots, A_n) \equiv \overline{A_i \mid \alpha_{ij}}$ under the distribution of shifts (4), the following system of functions of the algebra of logic:

* That is, the row in which the operator A_0 stands. In general, we shall not distinguish matrices that differ only in the order of arrangement of rows and columns; therefore, we shall always assign to rows and columns the numbers of the operators standing in them, independently of their arrangement in the matrix.

$$\alpha_i^0 = \alpha_{0i}, \qquad \beta_i^0 = \max_{\mathfrak{P}_i} \alpha_i^0 \quad (i = 1, 2, \dots, n),$$

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$$\alpha_i^{\nu+1} = \alpha_i^\nu \vee \bigvee_{j=1;\ j\neq i}^{n} \beta_j^\nu \alpha_{ji}, \qquad \beta_i^{\nu+1} = \max_{\mathfrak{P}_i} \alpha_i^{\nu+1} \quad (\nu = 0, 1, 2, \dots)$$

It is clear that as ν increases the functions α_i^ν and β_i^ν do not decrease, i.e. $\alpha_i^\nu \to \alpha_i^{\nu+1}$, $\beta_i^\nu \to \beta_i^{\nu+1}$; therefore there will be a μ such that for every $i = 1, 2, \dots, n$ one has $\alpha_i^\mu \equiv \alpha_i^{\mu+1}$, $\beta_i^\mu \equiv \beta_i^{\mu+1}$. The functions $\alpha_i^\mu, \beta_i^\mu$ for such μ will be denoted, respectively, by $A_i^*(\mathfrak{A}), A_i^{**}(\mathfrak{A})$. In addition, for every matrix, by definition, we put $A_0^*(\mathfrak{A}) \equiv A_0^{**}(\mathfrak{A}) \equiv 1$. Obviously, the functions $A_i^*(\mathfrak{A}), A_i^{**}(\mathfrak{A})$ so defined satisfy the equivalences

$$A_i^*(\mathfrak{A}) \equiv \bigvee_{j=0;\ j\neq i}^{n} A_j^{**}(\mathfrak{A}) \alpha_{ji}. \tag{5}$$

Theorem 1. In order that the matrix $\mathfrak{A}(p_1, \dots, p_k, A_1, \dots, A_n) \equiv \overline{A_i \mid \alpha_{ij}}$, under the given distribution of shifts, be an m.s., it is necessary and sufficient that for all $i = 0, 1, \dots, n;\ j, l = 1, 2, \dots, n;\ j \neq l$ the condition

$$A_i^*(\mathfrak{A}) \to (\alpha_{ij} \to \overline{\alpha}_{il}).$$

be satisfied.

* A distribution of shifts is a one-to-one correspondence established between the set of operators $A_1, \dots, A_n$ and a system of sets $\mathfrak{P}_i \subset \{p_1, \dots, p_k\}$ (1).

Definition 4. Matrix schemes $\mathfrak{A}, \mathfrak{B}$ will be called **equivalent** under a given distribution of shifts if, for every sequence of sets admissible for $\mathfrak{A}$ or $\mathfrak{B}$, their values coincide.

Theorem 2. In order that the matrix schemes

$$\mathfrak{A}(p_1, \dots, p_k, A_1, \dots, A_n) \equiv \overline{A_i \mid \alpha_{ij}}$$

and

$$\mathfrak{B}(p_1, \dots, p_k, A_1, \dots, A_n) \equiv \overline{A_i \mid \beta_{ij}}^{\,*}$$

be equivalent under a given distribution of shifts, it is necessary and sufficient that, for every $i = 0, 1, \dots, n$, the conditions

$$1) \quad A_i^{**}(\mathfrak{A}) \equiv A_i^{**}(\mathfrak{B}); \qquad 2) \quad A_i^{**}(\mathfrak{A}) \to (\alpha_{ij} \equiv \beta_{ij}) \quad (j = 1, 2, \dots, n)$$

be satisfied.

Let φ and α be functions of the algebra of logic. Denote by $\Pi_{\varphi(\alpha)}$ every function α' satisfying the condition:

$$\varphi \to (\alpha' \equiv \alpha).$$

We shall call the Π^i_φ-transformation of the matrix $\overline{A_i \mid \alpha_i}$ the replacement of an arbitrary element α_{ij} of its i-th row by the function

$$\alpha'_{ij} = \Pi_\varphi(\alpha_{ij}).$$

From (5) and Theorem 2, Theorem 3 follows easily.

Theorem 3. $\Pi^i_{A_i^{**}}$-transformations take a matrix scheme into an equivalent one, and conversely, every matrix scheme equivalent to a given one can be obtained from it by $\Pi^i_{A_i^{**}}$-transformations (not counting the addition or deletion of rows and columns all of whose elements are identically zero).

Definition 5. A set sequence $\mathfrak{A}$ and a matrix scheme $\overline{A_i \mid \alpha_{ij}}$ will be called **equivalent** under a given distribution of shifts if their values coincide for every sequence of sets admissible for $\mathfrak{A}$ or $\overline{A_i \mid \alpha_{ij}}$.

Definition 6. The matrix $\overline{A_i \mid \alpha_{ij}}$ will be called **complete** if, for every $i = 0, 1, \dots, n$,

$$A_i^{**} \to \bigvee_{j=1}^{n} \alpha_{ij}.$$

In order to be able to associate with every set sequence a certain matrix scheme, we shall consider, as an operator, the symbol of the empty period ( ) (^{1}), and also the symbol for the end of a process—the dot. This enables us to consider only complete matrix schemes. Indeed, for every matrix scheme (3) the matrix

$$\begin{array}{c|cccc} & A_1 & \cdots & A_n & \bullet \\ \hline A_0 & \alpha_{01} & \cdots & \alpha_{0n} & \alpha_{0\,n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ A_n & \alpha_{n1} & \cdots & \alpha_{nn} & \alpha_{n\,n+1} \end{array}$$

where

$$\alpha_{i\,n+1} \equiv \bigwedge_{j=1}^{n} \overline{\alpha}_{ij} \quad (i = 0, 1, \dots, n),$$

is complete and equivalent to the matrix scheme (3).

It is not difficult to construct an algorithm which, for every set sequence, gives a matrix scheme equivalent to it, and also an algorithm which, from a matrix scheme, gives a set sequence equivalent to it. Thus questions of equivalence of set sequences reduce to those for matrix schemes. Using this, it is not difficult to give a positive solution of the equivalence problem for set sequences in the presence of defining relations of the form $A_s = A_t$. In general, if there is a system of defining relations

$$A_{i_1^r} \cdots A_{i_{s_r}^r} = A_{j_1^r} \cdots A_{j_{t_r}^r} \qquad (r = 1, 2, \dots, n), \tag{6}$$

* Without loss of generality, we may assume that the matrix schemes under consideration have identical sets

then in the concept of equivalence of m.s. (s.d.) one should require, instead of coincidence of values, their equivalence in the associative calculus generated by the system (6)[3]*. For certain systems of defining relations the problem of equivalence of m.s. (and, consequently, of s.d.) has a negative solution. Thus, for example, if the system of defining relations is such that in the associative calculus generated by it the problem of equivalence to the empty word is unsolvable[3], then the problem of equivalence of m.s. (s.d.) under these defining relations has a negative solution.

Let us consider the case of one defining relation of the form $A_{k_1} = A_{k_2}$.

Let there be m.s.

$$\mathfrak{A}(A_1, \dots, A_n) \equiv \overline{A_i \mid \alpha_{ij}}, \qquad \mathfrak{B} \equiv \overline{A_i \mid \beta_{ij}}$$

and a distribution of shifts (4), where $\mathfrak{B}_{k_1} = \mathfrak{B}_{k_2}$. Denote

$$\chi_{ij}(\mathfrak{A}, \mathfrak{B}) = \bigvee_{l \neq k_1, k_2} \left(A_l^{**}(\mathfrak{A}) \alpha_{ij} \equiv A_l^{**}(\mathfrak{B}) \beta_{ij} \right),$$

where i, j independently take the values k_1, k_2. Suppose $\chi_{ij}^{\nu}(\mathfrak{A}, \mathfrak{B})$ are defined; then we put

$$\chi_{ij}^{\nu+1}(\mathfrak{A}, \mathfrak{B}) = \chi_{ij}^{\nu}(\mathfrak{A}, \mathfrak{B}) \vee \bigvee_{s,t = k_1, k_2} \left[\left(\max_{\mathfrak{B}_{k_1}} \chi_{st}^{\nu}(\mathfrak{A}\mathfrak{B}) \right) \alpha_{si} \beta_{tj} \right],$$

where the disjunction is taken over all s, t, independently taking the values k_1, k_2. Clearly, there exists such a μ that

$$\chi_{ij}^{\mu+1}(\mathfrak{A}, \mathfrak{B}) = \chi_{ij}^{\mu}(\mathfrak{A}, \mathfrak{B})$$

for all $i, j = k_1, k_2$. Denote $\chi_{ij}^{\mu}(\mathfrak{A}, \mathfrak{B})$ for such μ by χ_{ij}.

Theorem 4. For the m.s.

$$\mathfrak{A} \equiv \overline{A_i \mid \alpha_{ij}}, \qquad \mathfrak{B} \equiv \overline{A_i \mid \beta_{ij}}$$

to be equivalent under the condition $A_{k_1} = A_{k_2}$ (for a given distribution of shifts (4), where $\mathfrak{B}_{k_1} = \mathfrak{B}_{k_2}$), it is necessary and sufficient that the following conditions be fulfilled:

1) $\left[A_i^{**}(\mathfrak{A})(\alpha_{jk_1} \vee \alpha_{ik_2}) \equiv A_i^{**}(\mathfrak{B})(\beta_{ik_1} \vee \beta_{ik_2}) \right]_{j \neq k_1, k_2} \;\&\; \left(A_i^{**}(\mathfrak{A}) \alpha_{ij} \equiv A_i^{**}(\mathfrak{B}) \beta_{ij} \right),$

where $i \neq k_1, k_2$;

2) for $i, j = k_1, k_2$ $\quad \max\limits_{\mathfrak{B}_{k_1}} \chi_{ij} \to (\alpha_{ik_1} \vee \alpha_{ik_2} \equiv \beta_{jk_1} \vee \beta_{jk_2}) \mathbin{\&} \bigwedge\limits_{h \neq k_1, k_2} (\alpha_{ih} \equiv \beta_{jh})$.

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Received
2 X 1956

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* These concepts, however, require a certain extension, since one also has to consider words of infinite length.

Note: Figure translations are in progress. See original paper for figures.

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