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1957

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Abstract

Full Text

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DERIVATION OF AN EQUATION FOR THE INVERSE PROBLEM OF THE THEORY OF THE NEWTONIAN POTENTIAL

(Presented by Academician M. A. Lavrentiev, 5 IV 1957)

Under the inverse problem of the potential we understand the problem of finding a simply connected domain D, filled with matter of a given constant density, with a given external potential V_e. The function $\mathbf{r}(x, y, z) = X(x, y, z)\mathbf{i} + Y(x, y, z)\mathbf{j} + Z(x, y, z)\mathbf{k}$, mapping the sphere of unit radius with center at the origin of coordinates onto the domain D, will be called a solution of this inverse problem. In the present work, for three-dimensional space, a nonlinear integro-differential equation of the inverse problem of the potential is obtained, generalizing the integral equation of the inverse problem for two-dimensional space ([1]), under the assumption that the mapping function $\mathbf{r}(x, y, z)$ is potential-harmonic ([2]) outside the sphere being mapped. A vector function $\vec{\varphi}(x, y, z)$ is called potential-harmonic in a domain T if in this domain $\operatorname{div} \vec{\varphi}(x, y, z) = 0$ and $\operatorname{rot} \vec{\varphi}(x, y, z) = 0$.

If $\vec{\varphi}(x, y, z)$ is a potential-harmonic function in a domain containing $T + \sigma$; ρ is the distance between the points $M(\xi, \eta, \zeta)$ and $M_1(x, y, z)$; and $\mathbf{n}$ is the unit vector of the exterior normal to σ, then

$$\frac{1}{4\pi} \iint_\sigma \left[\operatorname{grad} \frac{1}{\rho(\xi, \eta, \zeta, x, y, z)} \, \mathbf{n}(\xi, \eta, \zeta) \, \vec{\varphi}(\xi, \eta, \zeta) \right] d\sigma = \begin{cases} \vec{\varphi}(x, y, z), & (x, y, z) \in T, \\ 0, & (x, y, z) \in T', \end{cases} \tag{1}$$

where T' is the domain complementing $T + \sigma$ to the whole space; under the integral sign stands the triple product of vectors defined in ([3]): $[abc] = -(b \cdot c)a + (c \cdot a)b - (a \cdot b)c$.

This formula was given by A. V. Bitsadze ([2]), but in another form. Using the notion of the triple product of vectors, we transform other formulas related to the theory of the Cauchy-type integral, given by A. V. Bitsadze ([2]). If the vector $\vec{\varphi}(x, y, z)$ is given only on the surface σ and its components satisfy the condition $H(\alpha)$, $0 < \alpha \leq 1$, then the integral analogous to the Cauchy-type integral has the form

$$\Phi(x, y, z) = \frac{1}{4\pi} \iint_\sigma \left[\operatorname{grad} \frac{1}{\rho(\xi, \eta, \zeta, x, y, z)} \, \mathbf{n}(\xi, \eta, \zeta) \, \vec{\varphi}(\xi, \eta, \zeta) \right] d\sigma. \tag{2}$$

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If the point (x_0, y_0, z_0) belongs to the surface σ, then, denoting by $\vec{\Phi}^+(x_0, y_0, z_0)$ and $\vec{\Phi}^-(x_0, y_0, z_0)$ the limiting values of $\vec{\Phi}(x, y, z)$ when the point (x, y, z) approaches (x_0, y_0, z_0), remaining, respectively, inside or outside T, we shall have:

$$\vec{\Phi}^+(x_0, y_0, z_0) = \frac{1}{2}\,\vec{\varphi}(x_0, y_0, z_0) + \\ + \frac{1}{4\pi} \iint_\sigma \left[\operatorname{grad} \frac{1}{\rho(\xi, \eta, \zeta, x_0, y_0, z_0)}\, \mathbf{n}(\xi, \eta, \zeta)\, \vec{\varphi}(\xi, \eta, \zeta)\right] d\sigma; \tag{3}$$

$$\vec{\Phi}^-(x_0, y_0, z_0) = -\frac{1}{2}\,\vec{\varphi}(x_0, y_0, z_0) + \\ + \frac{1}{4\pi} \iint_\sigma \left[\operatorname{grad} \frac{1}{\rho(\xi, \eta, \zeta, x_0, y_0, z_0)}\, \mathbf{n}(\xi, \eta, \zeta)\, \vec{\varphi}(\xi, \eta, \zeta)\right] d\sigma. \tag{4}$$

The integrals appearing on the right-hand sides exist in the sense of the principal value.

Let an exterior potential V_e be given, produced by matter filling, with a given constant density (we shall take it equal to unity), some sought region D of three-dimensional space, which is considered finite, simply connected, containing the origin in its interior and having a piecewise smooth boundary S. Denote by D' the region that completes $D + S$ to the whole space. Let $\mathbf{U}_e = \operatorname{grad} V_e$.

Obviously,

$$V_i = -\frac{2}{3}\pi(x^2 + y^2 + z^2) + W(x, y, z), \tag{5}$$

where $W(x, y, z)$ is a function harmonic in D.

On the boundary S of the region D one has

$$\operatorname{grad} V_i = \operatorname{grad} V_e,$$

or

$$-\frac{4}{3}\pi\mathbf{r} + \operatorname{grad} W = \mathbf{U}_e, \tag{6}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Using (1), we obtain

$$\frac{1}{4\pi} \iint_S \left[\operatorname{grad} \frac{1}{\rho(x, y, z, \xi, \eta, \zeta)}\, \mathbf{n}(x, y, z)\, \frac{4}{3}\pi\mathbf{r}(x, y, z)\right] dS =$$

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$$= \mathbf{U}_e(\xi,\eta,\zeta), \qquad (\xi,\eta,\zeta) \in D'. \tag{7}$$

If $M(x_0, y_0, z_0) \in S$, then, taking (4) into account, we obtain from (7):

$$\mathbf{U}_e(x_0,y_0,z_0) = -\frac{1}{2}\left(\frac{4}{3}\pi\mathbf{r}(x_0,y_0,z_0)\right) + \\ +\frac{1}{4\pi}\iint_S \left[\operatorname{grad}\frac{1}{\rho(x,y,z,x_0,y_0,z_0)}\,\mathbf{n}(x,y,z)\,\frac{4}{3}\pi\mathbf{r}(x,y,z)\right] dS. \tag{8}$$

Let the function $\mathbf{r}(x,y,z) = X(x,y,z)\mathbf{i} + Y(x,y,z)\mathbf{j} + Z(x,y,z)\mathbf{k}$, potential-harmonic in a region containing $T' + \sigma$, where T' is the region completing to all space the unit sphere T with center at the origin and with boundary σ, effect a mapping of the unit sphere T with center at the origin onto the sought region D; moreover, $\mathbf{r}$ satisfies the condition $H(\alpha)$, $0 < \alpha \le 1$.

If we assume that $\mathbf{r}(x,y,z) = X(x,y,z)\mathbf{i}+Y(x,y,z)\mathbf{j}+Z(x,y,z)\mathbf{k} = X_1(\theta,\varphi)\mathbf{i}+Y_1(\theta,\varphi)\mathbf{j}+Z_1(\theta,\varphi)\mathbf{k}$, where θ and φ are the spherical coordinates of a point on the unit sphere T, and the point $M'(x',y',z')$ is the point on T to which the point (x_0,y_0,z_0) on the surface S corresponds, then (8) can be represented in the form

$$\begin{aligned}
&U_e\left[X(x',y',z'),\ Y(x',y',z'),\ Z(x',y',z')\right] = \\
&= -\frac{4}{6}\pi\mathbf{r}\left[X(x',y',z'),\ Y(x',y',z'),\ Z(x',y',z')\right] + \\
&\quad + \frac{1}{4\pi}\iint_\sigma \left[\operatorname{grad}\frac{1}{\rho(\overline{M''}M')}\,[\mathbf{r}_\theta\mathbf{r}_\varphi]\,\frac{4}{3}\pi\mathbf{r}\left(X_1(\theta,\varphi),Y_1(\theta,\varphi),Z_1(\theta,\varphi)\right)\right]\frac{d\sigma}{\sin\theta} - \\
&\quad - \frac{1}{4\pi}\iint_\sigma \left[\operatorname{grad}\frac{1}{\rho(MM')}\,\mathbf{n}\,\frac{4}{3}\pi\mathbf{r}\left(X_1(\theta,\varphi),Y_1(\theta,\varphi),Z_1(\theta,\varphi)\right)\right] d\sigma + \\
&\quad + \frac{1}{4\pi}\iint_\sigma \left[\operatorname{grad}\frac{1}{\rho(MM')}\,\mathbf{n}\,\frac{4}{3}\pi\mathbf{r}\left(X_1(\theta,\varphi),Y_1(\theta,\varphi),Z_1(\theta,\varphi)\right)\right] d\sigma;
\end{aligned} \tag{9}$$

$\mathbf{n}$ is the unit vector of the outward normal to σ; $\overline{M''}$ is the point with coordinates $\{X_1(\theta,\varphi),Y_1(\theta,\varphi),Z_1(\theta,\varphi)\}$; M' is the point with coordinates $\{X(x',y',z'),Y(x',y',z'),Z(x',y',z')\}$; M is a variable point on σ. In view of (3), the first and the last terms on the right-hand side of (9) are equal.

Let

$$\frac{1}{4\pi}\iint_\sigma \left[\operatorname{grad}\frac{1}{\rho(\overline{M''}M')}\,[\mathbf{r}_\theta\mathbf{r}_\varphi]\,\frac{4}{3}\pi\mathbf{r}\left(X_1,Y_1,Z_1\right)\right]\frac{d\sigma}{\sin\theta} - \\ -\frac{1}{4\pi}\iint_\sigma \left[\operatorname{grad}\frac{1}{\rho(MM')}\,\mathbf{n}\,\frac{4}{3}\pi\mathbf{r}\left(X_1,Y_1,Z_1\right)\right] d\sigma = \mathbf{P}(x',y',z'),$$

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where the integrals are understood in the sense of the principal value.

Then, by virtue of (1), for $(\xi, \eta, \zeta) \in T'$ we have

$$\frac{1}{4\pi} \iint_\sigma \Big[\operatorname{grad} \frac{1}{\rho(x', y', z', \xi, \eta, \zeta)}\, \mathbf{n}(x', y', z')\, U_e(X(x', y', z'),$$

$$Y(x', y', z'), Z(x', y', z'))\Big]\, d\sigma = \tag{10}$$

$$= \frac{4}{3}\pi \mathbf{r}(\xi, \eta, \zeta) + \frac{1}{4\pi} \iint_\sigma \left[\operatorname{grad} \frac{1}{\rho(x', y', z', \xi, \eta, \zeta)}\, \mathbf{n}(x', y', z')\, \mathbf{P}(x', y', z')\right] d\sigma.$$

Thus, we have obtained the integro-differential equation (10) for the given inverse problem of potential. The function $\mathbf{r} = X(\xi, \eta, \zeta)\mathbf{i} + Y(\xi, \eta, \zeta)\mathbf{j} + Z(\xi, \eta, \zeta)\mathbf{k}$, mapping the unit sphere T with center at the origin onto the sought domain D, is the unknown.

For the two-dimensional case, the integral analogous to the last integral in (10) is equal to zero. This considerably simplifies the equation of the inverse problem of potential.

The equation obtained by us is easily generalized to the case of n-dimensional space.

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named after A. M. Gorky

Received
4 IV 1957

CITED LITERATURE

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[2] A. V. Bitsadze, *Izv. AN SSSR, ser. matem.*, **17**, 525 (1953).
[3] V. K. Ivanov, *Matem. sborn.*, **40** (82), 3 (1956).

Note: Figure translations are in progress. See original paper for figures.

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