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Abstract

Full Text

MATHEMATICS

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EXTREMAL PROBLEMS FOR DIFFERENTIABLE FUNCTIONALS IN THE THEORY OF UNIVALENT FUNCTIONS

(Presented by Academician V. I. Smirnov, 13 X 1956)

§ 1. Let G be some finitely connected domain, and let e be a closed set, $e \subset G$.

Denote by \mathfrak{M}_G the family of functions regular in the domain G , except possibly for a finite number of poles; by \mathfrak{R}_G , the family of functions regular in G ; and by K_e , some class of functions regular in G , except possibly for a finite number of poles in e .

Suppose that a certain real functional $\Phi[f]$ is defined on K_e . If there exists a function $f_0 \in K_e$ such that $\Phi[f] \leq \Phi[f_0]$ (maximum) or $\Phi[f] \geq \Phi[f_0]$ (minimum) for all $f \in K_e$, then the function f_0 will be called an **extremal function of the first kind** for the functional $\Phi[f]$ on K_e .

Suppose that functionals $\Phi_1[f], \dots, \Phi_n[f]$ ($n \geq 1$) (complex or real) are defined on K_e . Consider in the n -dimensional vector space C_n (real or complex) the vector $\vec{\Phi}[f]$, whose components are $\Phi_1[f], \dots, \Phi_n[f]$. We shall call $\vec{\Phi}[f]$ a functional*. Thus the functional $\vec{\Phi}[f]$ is defined on K_e . The set of elements of C_n which $\vec{\Phi}[f]$ assumes on the whole class K_e will be called the **range of variation** of $\vec{\Phi}[f]$ on K_e and denoted by $\{\vec{\Phi}[f]\}_{K_e}$ or $\{(\Phi_1[f], \dots, \Phi_n[f])\}_{K_e}$. We introduce in C_n all the usual definitions of set theory. If there exists a function $f_0 \in K_e$ such that $\vec{\Phi}[f_0]$ is a boundary vector of the range of variation $\{\vec{\Phi}[f]\}_{K_e}$, then the function f_0 will be called a **boundary function** or an **extremal function of the second kind**.

§ 2. Consider functionals defined on \mathfrak{M}_G , taking finite values on K_e , and weakly differentiable on K_e , i.e., such that for any pair of functions f and h ($f \in K_e$, $h \in \mathfrak{M}_G$) there exists a finite or infinite limit (with λ real)

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \{\Phi[f + \lambda h] - \Phi[f]\};$$

this limit, depending on f and h , is called the **functional derivative**.

We introduce the following definitions:

1. A real functional $\Phi[f]$ is called a **functional of type** A_{K_e} if its functional derivative is equal to $\operatorname{Re} D_f^{(\Phi)}[h]$, where $D_f^{(\Phi)}[h]$ is a complex functional, distributive with respect to $h \in \mathfrak{M}_G$, and assuming only finite values for $h \in \mathfrak{R}_G$.

* In the usual terminology, generally speaking, $\vec{\Phi}[f]$ is an operator.

2. A real functional of type A_{K_e} is called a functional of type $A_{K_e}^l$ if $D_f^{(\Phi)}[h]$ is continuous with respect to $h \in \mathfrak{A}_G^*$ and the function

$$\varphi(z) = D_f^{(\Phi)} \left[\frac{1}{\zeta - z} \right], \quad \zeta \in G,$$

is a rational function having poles only in G of order $\leq l$ and at least one pole of order l .

- 3-4. A complex functional is called a functional of type B_{K_e} (respectively of type $B_{K_e}^l$) if its functional derivative is equal to $D_f^{(\Phi)}[h]$, possessing the same properties as in 1 (respectively in 2).

§ 3. Let us give an important example of functionals of type $B_{K_e}^l$. Let

$$\Phi_k[f] = \sum_{\nu=1}^{p_k} a_{k\nu}^{(l_k)} f^{(l_k)}(z_{k\nu}), \quad z_{k\nu} \in G - e, \quad k = 1, \dots, n.$$

Then the functional

$$\Phi[f] = F(\Phi_1[f], \dots, \Phi_n[f])$$

is a functional of type $B_{K_e}^l$, if $F(w_1, \dots, w_n)$ is any function, defined for all values of its n complex variables, regular in an open domain containing $\{\Phi[f]\}_{K_e}$, and having in this domain a nonvanishing gradient.

In this case

$$D_f^{(\Phi)}[h] = \sum_{k=1}^n \alpha_k[f] \sum_{\nu=1}^{p_k} a_{k\nu}^{(l_k)} h^{(l_k)}(z_{k\nu}),$$

where

$$\alpha_k[f] = \left. \frac{\partial}{\partial w_k} F(w_1, \dots, w_n) \right|_{(w_1, \dots, w_n) = (\Phi_1[f], \dots, \Phi_n[f])}, \quad k = 1, \dots, n.$$

- § 4. By the variational method (1) one can investigate the properties of extremal functions of both the first and the second kind for the functionals defined in § 2, on various classes of univalent functions.

Take, for example, as G the unit disk $|z| < 1$, and as K_e the class S of functions $f(z) = z + c_2 z^2 + \dots$, regular and univalent in $|z| < 1$.

Theorem 1. If f is an extremal function of the first kind for a functional $\Phi[f]$ of type A_S on S , then f satisfies the equation

$$\begin{aligned} & \left(\frac{zf'(z)}{f(z)} \right)^2 D_f^{(\Phi)} \left[\frac{f(\zeta)^2}{f(\zeta) - f(z)} \right] = \\ & = \frac{1}{2} D_f^{(\Phi)} \left[f(\zeta) + \zeta f'(\zeta) \frac{\zeta + z}{\zeta - z} \right] + \frac{1}{2} D_f^{(\Phi)} \left[f(\zeta) + \zeta f'(\zeta) \frac{\zeta + 1/\bar{z}}{\zeta - 1/\bar{z}} \right]. \end{aligned} \quad (1)$$

The right-hand side of (1) is real on $|z| = 1$ and ≤ 0 (in the case of a maximum) or ≥ 0 (in the case of a minimum).

Theorem 2. If, under the hypotheses of Theorem 1, the functional $\Phi[f]$ is of type A_S^l ($l > 2$ for the case when

$$\varphi(z) = D_f^{(\Phi)} \left[\frac{1}{\zeta - z} \right]$$

has a pole of order l only at $z = 0$), then f maps $|z| < 1$ onto the whole plane with slits along a finite number of analytic curves, which, under a suitable choice of the real parameter τ , are integral curves of the differential equation

$$\left(\frac{dw}{d\tau} \right)^2 \frac{1}{w^2} D_f^{(\Phi)} \left[\frac{f(\zeta)^2}{f(\zeta) - w} \right] = \pm 1 \quad (2)$$

(the plus sign on the right-hand side of (2) corresponds to the case of a maximum).

* The definition of continuity of a functional on \mathfrak{A}_G is the usual one; convergence of elements $h_n \in \mathfrak{A}_G$, $n = 1, 2, \dots$, is understood as uniform convergence of h_1, h_2, \dots inside the domain G .

§ 5. It is known (see, for example, (1)) that to each function $f(z) \in S$ mapping $|z| < 1$ onto the entire plane with a finite number of analytic slits one can associate a complex function $K(t)$, $|K(t)| = 1$, continuous for all t , $0 \leq t < \infty$, except for a finite number of points $t = t_1, \dots, t_m$, such that

$$f(z) = \lim_{t \rightarrow 0} e^t f(z, t),$$

where $f(z, t)$ is the solution of the differential equation

$$\frac{\partial f}{\partial t} = -f \frac{1 + K(t)f}{1 - K(t)f}$$

with the initial condition

$$f|_{t=0} = z.$$

Theorem 3. Under the hypotheses of Theorem 2, the extremal function $w = f(z)$ satisfies, for every t , $0 \leq t < \infty$, the equation

$$\left(\frac{zf'(z)}{f(z)} \right)^2 \left(\frac{f(z, t)}{zf'(z, t)} \right)^2 D_f^{(\Phi)} \left[\frac{f(\zeta)^2}{f(\zeta) - f(z)} \right] =$$

$$\begin{aligned}
 &= \frac{1}{2} D_f^{(\Phi)} \left[f(\zeta) + \frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{f(\zeta, t) + f(z, t)}{f(\zeta, t) - f(z, t)} \right] + \\
 &+ \frac{1}{2} D_f^{(\Phi)} \left[f(\zeta) + \frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{f(\zeta, t) + \frac{1}{f(z, t)}}{\frac{1}{f(\zeta, t) + \frac{1}{f(z, t)}}} \right]. \quad (3)
 \end{aligned}$$

For all t , the right-hand side of (3) is real and ≤ 0 (or, respectively, ≥ 0) for such z that $|f(z, t)| = 1$.

Moreover, the following relations hold:

$$\operatorname{Re} D_f^{(\Phi)} \left[f(\zeta) - \frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{1 + K(t)f(\zeta, t)}{1 - K(t)f(\zeta, t)} \right] = 0; \quad (4)$$

$$\operatorname{Re} D_f^{(\Phi)} \left[f(\zeta) - \frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{1 + xf(\zeta, t)}{1 - xf(\zeta, t)} \right] \begin{cases} \leq 0 & (\text{maximum}), \\ \geq 0 & (\text{minimum}), \end{cases} \quad (5)$$

where $0 \leq t < \infty$, x is any point of the disk $|x| < 1$. The function $K(t) = e^{i\vartheta(t)}$ has derivatives of all orders with respect to t , $0 \leq t < \infty$, $t \neq t_1, \dots, t_m$. In addition,

$$\frac{d\vartheta}{dt} = - \frac{\operatorname{Im} D_f^{(\Phi)} \left[\frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{2K(t)^2 f(\zeta, t)^2}{(1 - K(t)f(\zeta, t))^4} \right]}{\operatorname{Re} D_f^{(\Phi)} \left[\frac{f(\zeta, t)}{f'(\zeta, t)} f'(\zeta) \frac{K(t)f(\zeta, t)(1 + K(t)f(\zeta, t))}{(1 - K(t)f(\zeta, t))^3} \right]}. \quad (6)$$

§ 6. Analogous results are obtained also for extremal functions of the second kind.

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REFERENCES

¹ G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 1952.

Note: Figure translations are in progress. See original paper for figures.

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