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## Abstract

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## MATHEMATICS

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# EQUIVALENT SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

*(Presented by Academician I. G. Petrovskii on June 8, 1957)*

Consider a system of linear differential equations in partial derivatives of the form

$$\frac{\partial^{n_i} u_i(x,t)}{\partial t^{n_i}} = \sum_{j=1}^{N_1} \sum_{(m_s)} A^{ij}_{(m_s)} \frac{\partial^{m_0+\cdots+m_n} u_j(x,t)}{\partial t^{m_0} \partial x_1^{m_1} \ldots \partial x_n^{m_n}}, \qquad (1)$$

$$(i = 1, \ldots, N_1;\ x = (x_1, \ldots, x_n)),$$

where $\sum_{(m_s)}$ denotes summation over all possible sets of indices

$$(m_0, \ldots, m_n), \qquad \text{with} \quad m_0 < n_j, \quad m_k < M \left( k = 1, \ldots, n,\ \sum_{i=1}^{N_1} n_i = N \right).$$

By introducing additional unknown functions

$$u_{N_1+1}(x,t) = \frac{\partial u_1}{\partial t}, \ldots, u_{N_1+n_1-1} = \frac{\partial^{n_1-1} u_1}{\partial t^{n_1-1}}, \ldots, u_N = \frac{\partial^{n_{N_1}-1} u_{N_1}}{\partial t^{n_{N_1}-1}},$$

the system (1) can be reduced to the form

$$\frac{\partial u(x,t)}{\partial t} = P\left(i\frac{\partial}{\partial x}\right) u(x,t), \qquad (1')$$

where $u(x,t) = \{u_1(x,t), \ldots, u_N(x,t)\}$, $x = (x_1, \ldots, x_n)$; $P\left(i\frac{\partial}{\partial x}\right)$ is a square matrix of order $N$, whose elements are polynomials in

$$i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}.$$

**Definition 1.** System ($1'$) is called equivalent to the system

$$\frac{\partial v(x,t)}{\partial t} = Q\left(i\frac{\partial}{\partial x}\right) v(x,t), \tag{1”}$$

if both systems contain the same number of unknown functions and if there exists a nonsingular operator $T\left(i\frac{\partial}{\partial x}\right)$ (where $T\left(i\frac{\partial}{\partial x}\right)$ is a matrix of order $N$, $\det T(s) \not\equiv 0$) such that, if $u(x,t)$ is a solution of system ($1'$), then

$$v(x,t) = T\left(i\frac{\partial}{\partial x}\right) u(x,t)$$

is a solution of system ($1''$).

**Theorem 1.** *If system ($1'$) is equivalent to system ($1''$), then the matrix $Q(s)$ is similar to the matrix $P(s)$.*

**Proof.** Applying the operator $T\left(i\frac{\partial}{\partial x}\right)$ to both sides of ($1'$) and taking into account that the function

$$v(x,t) = T\left(i\frac{\partial}{\partial x}\right) u(x,t)$$

is a sol–

by the solution of system ($1''$), we obtain

$$T\left(i\frac{\partial}{\partial x}\right) P\left(i\frac{\partial}{\partial x}\right) u(x,t) = Q\left(i\frac{\partial}{\partial x}\right) T\left(i\frac{\partial}{\partial x}\right) u(x,t),$$

where $u(x,t)$ is any solution of system ($1'$). Passing to Fourier transforms, we obtain

$$T(s)P(s)\tilde{u}(s,t) = Q(s)T(s)\tilde{u}(s,t),$$

where $\tilde{u}(s,t)$ is any solution of the system

$$\frac{d\tilde{u}(s,t)}{dt} = P(s)\tilde{u}(s,t).$$

Choosing those solutions of the latter system which for $t=0$ coincide with the functions $\tilde{u}(s,0) = \{0,\dots,0,1,0,\dots,0\}$ and putting $t=0$, we obtain $T(s)P(s) = Q(s)T(s)$, as was required.

**Theorem 2.** *If the matrix $Q(s)$ is similar to the matrix $P(s)$, then system $(1')$ is equivalent to system $(1'')$.*

**Proof.** Let

$$Q(s) = T(s)P(s)T^{-1}(s), \tag{2}$$

where $\det T(s) \neq 0$, and the elements of the matrix $T(s)$ are polynomials in $s_1, \dots, s_n$ (otherwise, instead of $T(s)$ one may consider the matrix $T_1(s) = \tau(s)T(s)$, where $\tau(s)$ is the common denominator of the elements of the matrix $T(s)$).

Applying to both sides of $(1')$ the operator $T\left(i\frac{\partial}{\partial x}\right)$, denoting

$$v(x,t) = T\left(i\frac{\partial}{\partial x}\right)u(x,t),$$

and taking (2) into account, we obtain that, if $u(x,t)$ is a solution of system $(1')$, then $v(x,t)$ is a solution of system $(1'')$, as was required.

**Corollary 1.** *If system $(1')$ is equivalent to system $(1'')$, then system $(1'')$ is equivalent to system $(1')$.*

**Corollary 2.** *A necessary and sufficient condition for the equivalence of systems $(1')$ and $(1'')$ is the similarity of the matrices $P(s)$ and $Q(s)$.*

**Remark 1.** The solutions $u(x,t)$ and $v(x,t)$ of systems $(1')$ and $(1'')$ should be considered as generalized functions over some space of test functions, since, for example, the function $u(x,t)$ may fail to have a sufficient number of ordinary derivatives for the operator $T\left(i\frac{\partial}{\partial x}\right)$ to be applicable to it.

**Remark 2.** Suppose that systems $(1')$ and $(1'')$ are equivalent and that the equality $Q(s) = T(s)P(s)T^{-1}(s)$ holds, where $\det T(s) \neq 0$, and the elements of the matrix $T(s)$ are polynomials in $s_1, \dots, s_n$. Then, as an operator transforming solutions of system $(1')$ into solutions of system $(1'')$, one may take the operator $T\left(i\frac{\partial}{\partial x}\right)$. As an operator transforming solutions of system $(1'')$ into solutions of system $(1')$, one may take the operator $R\left(i\frac{\partial}{\partial x}\right)$, where

$$R(s) = \tau(s)T^{-1}(s),$$

and $\tau(s)$ is the common denominator of the elements of the matrix $T^{-1}(s)$.

**Theorem 3.** *If system $(1')$ is equivalent to system $(1'')$, then every solution $v(x,t)$ of system $(1'')$ is obtained as the result of applying a nonsingular operator $T\left(i\frac{\partial}{\partial x}\right)$ to some solution $u(x,t)$ of system $(1')$.*

**Proof.** Suppose that the nonsingular operator $T\left(i\frac{\partial}{\partial x}\right)$ takes every solution $u(x,t)$ of system $(1')$ into a solution $v(x,t)$ of system $(1'')$, and let $v_0(x,t)$ be some solution of system $(1'')$.

Consider the system

$$T\left(i\frac{\partial}{\partial x}\right)u(x,t)=v_0(x,t). \tag{3}$$

A solution $u(x,t)$ of this system exists by the Ehrenpreis theorem ([1]). We shall show that it satisfies system $(1')$. Applying the Fourier transform to both sides of (3), we obtain

$$T(s)\tilde{u}(s,t)=\tilde{v}_0(s,t),$$

where $\tilde{v}_0(s,t)$ is a solution of the system

$$\frac{d\tilde{v}_0(s,t)}{dt}=Q(s)\tilde{v}_0(s,t),$$

since $v_0(s,t)$ is a solution of system $(1'')$. Then

$$\frac{d\tilde{u}(s,t)}{dt}=T^{-1}(s)T(s)\frac{d\tilde{u}}{dt}=T^{-1}(s)\frac{d\tilde{v}_0(s,t)}{dt}=T^{-1}(s)Q(s)\tilde{v}_0(s,t)=$$

$$=T^{-1}(s)T(s)P(s)T^{-1}(s)T(s)\tilde{u}(s,t)=P(s)\tilde{u}(s,t).$$

Performing the inverse Fourier transform, we obtain that $u(x,t)$ is a solution of system $(1')$, as was required.

The equivalence of systems $(1')$ and $(1'')$ does not in general guarantee a one-to-one correspondence between the solutions of systems $(1')$ and $(1'')$. Such a correspondence exists, for example, when $\det T(s)=\text{const}$. Then, if the operator $T\left(i\frac{\partial}{\partial x}\right)$ maps a solution $u(x,t)$ of system $(1')$ to a solution $v(x,t)$ of system $(1'')$, then the operator

$$R\left(i\frac{\partial}{\partial x}\right)=T^{-1}\left(i\frac{\partial}{\partial x}\right)$$

maps the solution $v(x,t)$ of system $(1'')$ to the solution $u(x,t)$ of system $(1')$.

In the general case, the totalities of solutions of system $(1')$ and of system $(1'')$ may be divided into disjoint classes of equivalent solutions, between which a one-to-one correspondence is already established. This division into classes is carried out as follows.

Let $T\left(i\frac{\partial}{\partial x}\right)$ be an operator that maps every solution of system $(1')$ to a solution of system $(1'')$, and let the operator $R\left(i\frac{\partial}{\partial x}\right)$ map every solution of system $(1'')$ to a solution of system $(1')$, and let

$$\tau(s)E=T(s)R(s).$$

**Definition 2.** Two solutions $u_1(x,t)$ and $u_2(x,t)$ of system $(1')$ (or of system $(1'')$) are called **equivalent** if, for some integers $m_1 \geq 0$ and $m_2 \geq 0$, the equality

$$\tau^{m_1}\left(i\frac{\partial}{\partial x}\right) E \cdot u_1(x,t) = \tau^{m_2}\left(i\frac{\partial}{\partial x}\right) E \cdot u_2(x,t)$$

holds.

If $U$ is a class of equivalent solutions of system $(1')$, then, as is not hard to establish,

$$T\left(i\frac{\partial}{\partial x}\right) U = V,$$

where $V$ is a class of equivalent solutions of system $(1'')$. Then

$$R\left(i\frac{\partial}{\partial x}\right) V = \tau\left(i\frac{\partial}{\partial x}\right) U = U.$$

Thus, between the classes of equivalent solutions of systems $(1')$ and $(1'')$ a one-to-one correspondence is established.

**Definition 3.** Two systems of the form (1) are called **equivalent** if the corresponding systems of the form $(1')$ are equivalent.

It was established in ([2]) that any system of the form (1) is equivalent to one partial differential equation with constant coefficients of the form

$$\frac{\partial^N u(x,t)}{\partial t^N} = \sum_{m=1}^{N} P_m\left(i\frac{\partial}{\partial x}\right)\frac{\partial^{N-m}u(x,t)}{\partial t^{N-m}}$$

or to a system of several equations of the form

$$\frac{\partial^{N_k} u_k(x,t)}{\partial t^{N_k}} = \sum_{m=1}^{N_k} P_{mk}\left(i\frac{\partial}{\partial x}\right)\frac{\partial^{N_k-m}u_k(x,t)}{\partial t^{N_k-m}} \tag{4}$$

$$\left(k = 1, \dots, p,\ \sum_{k=1}^{p} N_k = N\right),$$

each of which is integrated independently of the others.

**Remark.** In reducing system (1) to system $(1')$, the characteristic roots do not change. Therefore, by virtue of Corollary 2, for equivalent systems of the form (1) the characteristic roots (with their multiplicities) coincide and, consequently, properties of systems based only on properties of the characteristic roots coincide for equivalent systems (for example, the belonging of a system to the hyperbolic, parabolic, Petrovskii-correct, and analytically correct types; for definitions see ([3])).

**Theorem 4.** *If system (1) is elliptic*, then each of the equations of system (4), to which it is equivalent, is also elliptic.*

**Example.** The system**

$$\begin{aligned} &\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} - 2\frac{\partial^2 u_2}{\partial x\, \partial y} = 0, \\ &2\frac{\partial^2 u_1}{\partial x\, \partial y} + \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial x^2} = 0 \end{aligned} \tag{5}$$

is equivalent to the biharmonic equation $\Delta\Delta v = 0$, where

$$T(s) = \begin{Vmatrix} 0 & 2is & -1 & 0 \\ s^2 & 0 & 0 & 0 \\ 0 & 0 & s^2 & 0 \\ -s^4 & 0 & 0 & 2is^3 \end{Vmatrix}, \qquad R(s) = \begin{Vmatrix} 0 & 2is & 0 & 0 \\ s^2 & 0 & 1 & 0 \\ 0 & 0 & 2is & 0 \\ 0 & s^2 & 0 & 1 \end{Vmatrix},$$

and $\tau\left(i\frac{\partial}{\partial y}\right) = -2\frac{\partial^3}{\partial y^3}$. A solution $u(x, y)$ of system (5) is equivalent to a solution $v(x, y)$ of this system if, for some $m \geq 0$,

$$u_i = (-2)^{3m}\frac{\partial^{3m} v_i}{\partial y^{3m}} + \sum_{k=0}^{3m} C_{ki}(x) y^k \qquad (i = 1, 2, 3, 4)$$

(here $u(x, y) = \{u_1, u_2, u_3, u_4\}$, $v(x, y) = \{v_1, v_2, v_3, v_4\}$, $u_1 = u_1(x, y)$, $u_2 = \dfrac{\partial u_1}{\partial x}$, $u_3 = u_2(x, y)$, $u_4 = \dfrac{\partial u_2(x, y)}{\partial x}$; the components of the solution $v(x, y)$ are defined analogously).

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* For the definition see ([4]).
** This system is given in ([5]) as an example of an elliptic system for which the Dirichlet problem is not well posed.