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**Abstract**

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# CRITICAL CONDITIONS FOR INERTIAL DEPOSITION OF AEROSOLS IN VISCOUS FLOW AROUND A CYLINDER AND A SPHERE

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The equations of motion of an aerosol particle in a flow, for Stokes' law of resistance of the medium, have the form

$$m\frac{du_i}{dt} = 6\pi\eta R\,(v_i - u_i), \tag{1}$$

where $m$ and $R$ are the mass and radius of the particle, $\eta$ is the viscosity of the medium, $t$ is time, and $v_i$ and $u_i$ are the components of the velocities of the flow and of the particle along the coordinate axis $i$. Owing to inertia, aerosol particles at places where the flow turns deviate from the streamlines. When a flow passes around an obstacle, aerosol particles, deviating from the streamlines, may reach the surface of the obstacle and thus be deposited on it. If the velocity and position of the aerosol particle at the initial instant of time at a large distance from the obstacle are known, then on the basis of equations (1), knowing $v_i$ as functions of the coordinates, it is in principle possible to compute the trajectory of motion of the given particle and thereby determine whether it will be captured by the obstacle.

Introducing dimensionless velocities and time $V_i = \frac{v_i}{v_0}$, $U_i = \frac{u_i}{v_0}$, and $\tau = \frac{v_0 t}{a}$, where $v_0$ is the velocity of the incident undisturbed flow and $a$ is the linear size of the obstacle, the system of equations (1) can be written in the form

$$P\frac{dU_i}{d\tau} = V_i - U_i, \tag{2}$$

where $P = \frac{2R^2\gamma_1 v_0}{9\eta a}$ and $\gamma_1$ is the particle density. Let, far from the obstacle, the particle move along a streamline with a velocity coinciding with the velocity of the flow. For a given initial position of the particle its trajectory will be completely determined by the velocity field of the flow and by the magnitude of the dimensionless parameter $P$, often called the Stokes number. The value of the capture coefficient $\varepsilon = \frac{s_1}{s_2}$, where $s_1$ is the cross section of the stream tube of the undisturbed incident flow bounded by the limiting trajectories of particles still captured by the obstacle, and $s_2$ is the midsection of the obstacle, is also completely determined by the value of $P$ and by the velocity field of the flow. As $P$ decreases, the limiting trajectories of particles still captured by the obstacle approach the central streamline and the value of $\varepsilon$ decreases. For each velocity field there exists a critical value $P = P_{\text{cr}}$, below which $\varepsilon = 0$.

The values of $P_{\text{cr}}$ for potential flow in the case of plane-symmetric and axially symmetric problems were determined analytically by L. M. Levin ([1]), who showed that under these conditions

$$\frac{1}{P_{\text{cr}}} = -4\left(\frac{\partial V_x}{\partial X}\right)_{X=-1,\, Y=0,\, Z=0}, \tag{3}$$

where $X = \frac{x}{a}$, $Y = \frac{y}{a}$, and $Z = \frac{z}{a}$ are dimensionless coordinates; here the origin is at the center of the obstacle, and the undisturbed flow moves in the direction of increasing $x$. Davies and Peetz ([2]) (without sufficient grounds) used expression (3) to calculate $P_{\text{cr}}$ for viscous flow past a cylinder. Meanwhile, from the boundary conditions for viscous flow (on the surface of the obstacle everywhere $v_i = 0$) it follows that, when the central streamline is normal to the surface of the obstacle, at the critical point ($X = -1$, $Y = 0$, $Z = 0$)

$$\frac{\partial v_y}{\partial y} = \frac{\partial v_z}{\partial z} = 0,$$

whence, taking into account the incompressibility equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0,$$

one obtains for the critical point

$$\frac{\partial v_x}{\partial x} = 0.$$

Thus, when (3) is applied to viscous flow, one must always obtain $P_{\text{cr}} = \infty$. Davies and Peetz obtained finite values for $P_{\text{cr}}$ in their work; however, this is due only to the fact that they used approximate expressions for the components of the flow velocity which do not exactly satisfy the boundary conditions for viscous flow. Thus, the values of $P_{\text{cr}}$ calculated by them have no real meaning and are determined entirely by the degree to which the approximate value of the flow-velocity gradient along the central axis at the critical point deviates from its exact zero value.

Real values of the quantity $P_{\text{cr}}$ for viscous flow can be obtained by numerical integration of equations (2). For this it is necessary to know $v_i$ as functions of the coordinates. Expressions for the components of the flow velocity in viscous flow, in a perpendicular stream, past an infinite circular fixed cylinder, suitable at any distances from the cylinder, can be obtained from the Oseen equations ([3]):

$$v_x = \frac{\partial \varphi}{\partial x} + \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi, \qquad v_y = \frac{\partial \varphi}{\partial y} + \frac{1}{2k}\frac{\partial \chi}{\partial y}, \tag{4}$$

where

$$\varphi = \sum_{n=0}^{\infty} A_n \frac{\partial^n \ln r}{\partial x^n}, \qquad \chi = -v_0 + e^{kx}\sum_{n=0}^{\infty} B_n \frac{\partial^n K_0(kr)}{\partial x^n},$$

$K_0$ is a Bessel function of imaginary argument, $k = \dfrac{v_0\gamma_2}{2\eta}$, $\gamma_2$ is the density of the medium, and $r$ is the polar radius vector. Since the constants $A_n$ have magnitudes of order $(ka)^{n-1}$, and $B_n$ of order $(ka)^n$, at small Reynolds numbers $\mathrm{Re} = 4ka$ one may restrict oneself to the initial terms of the expansions of $\varphi$ and $\chi$ in series. Restricting ourselves in the expressions for $v_i$ to terms of order no higher than $ka$, it is sufficient to take three terms in the expansion for $\varphi$ and three terms in the expansion for $\chi$ (in fact it is sufficient to take two terms in the expansion for $\chi$, since, when terms of order $k^2a^2$ are neglected, the constants $A_2$ and $B_2$ enter the expressions for $v_i$ in the linear combination $A_2 - \dfrac{B_2}{2k} = A_2^0$). Substituting the expressions for $\varphi$ and $\chi$ thus obtained into equations (4), neglecting terms of order higher than $ka$, and determining the values of the constants $A_0$, $A_1$, $A_2^0$, $B_0$, and $B_1$ from the boundary conditions $v_x = v_y = 0$ at $r = a$ by equating to zero the terms with different powers of $x$ and $y$, we obtain (in what follows $K_0(ka)$ is denoted by $K_0$):

$$\begin{aligned} v_x = v_0 - \frac{2v_0}{1+2K_0}\Bigg\{ & e^{kx}K_0(kr) + \left(e^{kx}K_1(kr) - \frac{1}{kr}\right) \\ & \times \left[\frac{x}{r} + \frac{ka^2}{r}\left(\frac{1}{4}+K_0\right)\left(1 - 2\frac{x^2}{r^2}\right)\right] + \frac{1}{2}\frac{a^2}{r^2} - \frac{a^2x^2}{r^4} - \frac{3ka^4xK_0}{2r^4} + \frac{2ka^4x^3K_0}{r^6}\Bigg\}, \end{aligned} \tag{5}$$

$$\begin{aligned} v_y = \frac{2v_0}{1+2K_0}\Bigg\{ & \left(e^{kx}K_1(kr) - \frac{1}{kr}\right)\left[\frac{2ka^2xy}{r^3}\left(\frac{1}{4}+K_0\right) - \frac{y}{r}\right] \\ & + \frac{a^2xy}{r^4} - \frac{ka^2y}{r^2}\left(\frac{1}{4}+K_0\right) + \frac{ka^4yK_0}{2r^4} - \frac{2ka^4x^2yK_0}{r^6}\Bigg\}. \end{aligned}$$

If terms of order $ka$ and $kr$ are neglected in (5), the well-known expressions of Lamb are obtained.

Expressions (5) differ in the first-order terms with respect to $ka$ from the expressions given by Davies[4] and used by him[2,5] for calculating $\varepsilon$ as a function of $P$ at $\mathrm{Re} = 0.2$. In calculating $v_x$ and $v_y$ from equations (4), Davies used five terms in the expansion of $\varphi$ and only one term in the expansion of $\chi$. To determine the six constants, from the boundary conditions he obtained seven inconsistent equations and was able to find the values of the constants and satisfy the boundary conditions only by omitting, without sufficient grounds, a number of terms. Therefore the expressions he obtained for $v_i$ do not satisfy equations (4) in terms of order $ka$.

To determine the value $P_{\mathrm{cr}}$, it is not necessary to find the values of $\varepsilon$ as a function of $P$; it is sufficient to consider only the motion of an aerosol particle along the central streamline and to determine at what minimum value $P = P_{\mathrm{cr}}$ the particle will still reach the surface of the obstacle.

In what follows we shall denote the dimensionless velocities of the flow and of the particle along the central axis by the letters $V$ and $U$, without subscripts. Then the equation of motion of an aerosol particle along the central $x$-axis takes the form

$$P\frac{dU(x)}{d\tau} = V(x) - U(x). \tag{6}$$

The problem of determining $P_{\mathrm{cr}}$ reduces to determining the minimum value of $P$ for which, at the surface of the obstacle, i.e. at $X = -1$, $U > 0$ $(U - V > 0)$.

One of the most accurate and convenient methods of numerical integration of differential equations is Adams' method[6]. To make it possible to apply this method, we first transform equation (6) by means of $U\,d\tau = dX$ into $PU\,dU/dX = V - U$, and then, introducing the variable $h = -1/X$, into

$$\frac{dU}{dh} = \frac{V - U}{PUh^2}. \tag{7}$$

To determine $P_{\mathrm{cr}}$, it is necessary, by means of this equation, to find for various $P$ the values of $U$ at $h = 1$ under the initial condition: $U = 1$ at $h = 0$.

The dependence of $V$ on $h$ for the incident flow is given, on the basis of (5), by the following expression ($r = -x = a/h$):

$$V = 1 - \frac{2}{1 + 2K_0}\left\{e^{-ka/h}K_0(ka/h) - \left(e^{-ka/h}K_1(ka/h) - \frac{h}{ka}\right)\times\right.$$
$$\left.\times\left(1 + \frac{1}{4}hka + hkaK_0\right) - \frac{1}{2}h^2 - \frac{1}{2}h^3kaK_0\right\}. \tag{8}$$

For numerical integration of (7) by Adams' method, it is first necessary to obtain, in some way, three or four initial values of $U$ for equal small intervals $\Delta h$. Therefore equation (7) was first integrated for small values of $h$ $\left(V = 1 - \dfrac{2h}{ka(1+2K_0)} = 1 - Ch\right)$ by the method of undetermined coefficients by expanding $U$ in a series: $U = 1 + b_1 h + b_2 h^2 + b_3 h^3 + \cdots$. Substitution of these expressions for $V$ and $U$ into (7) and equating coefficients of like powers of $h$ gives

$$U = 1 - Ch + CPh^2 - CP(C+2P)h^3 + 6CP^2(C+P)h^4 - \cdots. \tag{9}$$

The determination of $P_{\mathrm{cr}}$ was carried out for $\mathrm{Re} = 4ka = 0.1$. First, by means of (9), the values of $U$ were calculated for $h \leq 85 \cdot 10^{-4}$. Then, by Adams' method, the values of $U$ for $h > 85 \cdot 10^{-4}$ were calculated over intervals $\Delta h$, which at first were $2.5 \cdot 10^{-4}$, and then were increased up to the value $\Delta h = 4 \cdot 10^{-2}$. The intervals were chosen so that in the calculation it was possible to ensure sufficient accuracy of the value $\Delta h\, dU/dh$ and so that it was possible to restrict oneself to second (very rarely third) differences of this quantity. The whole region from $h = 85 \cdot 10^{-4}$ to $h = 1$ was divided into 94 intervals. The calculation of $V$ and $U$ pro-

was carried out with accuracy to the fourth digit. In computing $V$, eight-place tables of the functions $K_0$ and $K_1$ with argument interval 0.001 [1] were used. For intermediate values of the argument, linear interpolation, or, when necessary, parabolic interpolation, was used. The computation of $U$ was carried out for a number of values of $P$. As the critical value, above which for $h = 1$, $U - V > 0$, at $\mathrm{Re} = 0.1$ the value $P_{\mathrm{cr}} = 4.3 \pm 0.1$ was obtained (for potential flow past a cylinder $P_{\mathrm{cr}} = 1/8$ [2,3]).

By an analogous method, $P_{\mathrm{cr}}$ was also computed for Oseen viscous flow past a sphere. In this case the following expressions are obtained for the components of the flow velocity (to terms of order $ka$):

[1] *Tables of Values of Bessel Functions of Imaginary Argument*, Publishing House of the Academy of Sciences of the USSR, 1950.

[2] L. M. Levin, *DAN*, **91**, 1329 (1953).

[3] I. Langmuir, *J. Meteorol.*, **5**, 175 (1948), trans. in the collection *Physics of the Formation of Precipitates*, IL, 1951.

$$v_x = v_0 - \frac{3v_0}{4-3ka}\frac{a}{r}\left[e^{k(x-r)}\left(1+\frac{x}{r}+\frac{x}{kr^2}+\frac{a^2}{r^2}-3\frac{a^2x^2}{r^4}\right)-\frac{x}{kr^2}\right.$$
$$\left.-\frac{2a^2}{3r^2}+2\frac{a^2x^2}{r^4}+\frac{ka^2}{r}\left(1-\frac{3x^2}{r^2}-\frac{3a^2x}{2r^3}+\frac{5a^2x^3}{2r^5}\right)\right],$$
$$v_\rho = \frac{3v_0}{4-3ka}\frac{a\rho}{r^2}\left[e^{k(x-r)}\left(3\frac{a^2x}{r^3}-\frac{1}{kr}-1\right)+\frac{1}{kr}-2\frac{a^2x}{r^3}\right.$$
$$\left.-\frac{ka^2}{r}\left(1-\frac{3x}{r}-\frac{a^2}{2r^2}+\frac{5a^2x^2}{2r^4}\right)\right]. \qquad (10)$$

Here $x$ and $\rho$ are cylindrical coordinates, and $r$ is the spherical radius vector. For $ka \to 0$ and $kr \to 0$, these expressions pass into the expressions for Stokes flow. The computation gave, at $\mathrm{Re} = 0.1$, the value $P_{\mathrm{cr}} = 1.15 \pm 0.01$.

Finally, an analogous computation was carried out for Stokes flow past a sphere. In this case the value $P_{\mathrm{cr}} = 1.21 \pm 0.01$ was found, in complete agreement with the value $P_{\mathrm{cr}} = 1.214$ obtained by Langmuir [4] with the aid of a differential analyzer (for potential flow past a sphere $P_{\mathrm{cr}} = 1/12$ [5,6]).

It should be emphasized that the concept of $P_{\mathrm{cr}}$ applies only to the purely inertial part of the deposition process. The conclusion that $\varepsilon = 0$ for $P < P_{\mathrm{cr}}$, as is seen from the preceding, is obtained only as a consequence of the assumption that capture of a particle occurs when its center of gravity reaches the surface of the obstacle. In fact, capture occurs when the particle approaches the surface of the obstacle to a distance equal to its radius. If this circumstance is taken into account, then even for $P < P_{\mathrm{cr}}$ capture will occur, caused by the pure effect of interception (contact). For viscous flow past a cylinder, for small $\varepsilon$ the capture coefficient due to interception is

$$\varepsilon = \frac{2}{1+2K_0}\frac{R^2}{a^2}.$$

Thus, the influence of particle size on capture will be small only when $R \ll a$. The latter condition, strictly speaking, must also be satisfied for the possibility of neglecting the action of the particle on the velocity field of the flow past the obstacle. Since

[4] I. Langmuir, *J. Meteorol.*, **5**, 175 (1948), trans. in the collection *Physics of the Formation of Precipitates*, IL, 1951.
[5] L. M. Levin, *DAN*, **91**, 1329 (1953).
[6] I. Langmuir, *J. Meteorol.*, **5**, 175 (1948), trans. in the collection *Physics of the Formation of Precipitates*, IL, 1951.

$$P = \frac{R^2 \gamma_1 \mathrm{Re}}{9a^2 \gamma_2},$$

then for $P > P_{\mathrm{cr}}$,

$$\frac{R^2}{a^2} > \frac{9\gamma_2 P_{\mathrm{cr}}}{\gamma_1 \mathrm{Re}}.$$

Thus, at very small Re, simultaneous fulfillment of the conditions $P > P_{\mathrm{cr}}$ and $R \ll a$ is impossible. At $\mathrm{Re} = 0.1$, $P_{\mathrm{cr}} = 4.3$ and $\gamma_2 = 1.2 \cdot 10^{-3}$, the condition $P > P_{\mathrm{cr}}$ gives

$$\frac{R^2}{a^2} > \frac{0.46}{\gamma_1},$$

i.e., fulfillment of the condition $R \ll a$ is then possible only for aerosols of certain metals ($\gamma_1 \gg 1$).

Physicochemical Institute
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## REFERENCES

*Note: Figure translations are in progress. See original paper for figures.*

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