MATHEMATICS

A. F. TIMAN and M. F. TIMAN

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Abstract

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ON THE DEPENDENCE BETWEEN MODULI OF SMOOTHNESS OF FUNCTIONS DEFINED ON THE WHOLE REAL AXIS

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Let $1 \leqslant p \leqslant \infty$, and let $f(x)$ be an arbitrary measurable function defined on the interval $(-\infty, \infty)$, for which*

$$\|f\|_{L_p} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx \right)^{1/p} < \infty. \tag{1}$$

For any natural $k \geqslant 1$, consider the function

$$\omega_k(f,t)_{L_p} = \sup_{|h| \leqslant t} \left[\int_{-\infty}^{\infty} \left| \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu h) \right|^p dx \right]^{1/p}, \tag{2}$$

defined on the half-axis $t \geqslant 0$ and representing the modulus of smoothness of order k for $f(x)$ in the corresponding metric. It is evident that if $k < \nu$, then

$$\omega_\nu(f;t)_{L_p} \leqslant 2^{\nu-k} \omega_k(f;t)_{L_p}. \tag{3}$$

One can give examples of functions for which this inequality, giving an upper estimate of moduli of smoothness through moduli of smoothness of lower orders, becomes, with respect to order (as $t \to 0$), an equality.

The following proposition seems to us of interest; it makes it possible to estimate from above the order of the moduli of smoothness of a function by means of its moduli of smoothness of higher orders.

Theorem. If $1 \leqslant k < \nu$, then for $0 < t \leqslant \frac{1}{2}$

$$\omega_k(f;t)_{L_p} \leqslant C_{\nu,k} t^k \int_t^1 \int_{t_1}^2 \cdots \int_{t_{\nu-k-1}}^2 \frac{\omega_\nu(f;t_{\nu-k})_{L_p}}{t_{\nu-k}^{\nu}} \, dt_1 \cdots dt_{\nu-k}, \tag{4}$$

where $C_{\nu,k}$ is a constant independent of the function f. In particular, for $k \geqslant 1$ the inequality

$$\omega_k(f;t)_{L_p} \leqslant C_k t^k \int_t^1 \frac{\omega_{k+1}(f;u)_{L_p}}{u^{k+1}}\, du. \tag{5}$$

always holds.

* In the case $p = \infty$ we put

$$\|f\|_{L_\infty} = \operatorname{vrai\,sup}_{-\infty<x<\infty} |f(x)|.$$

From inequality (4) it follows directly that

Corollary 1. If $\nu > k$, $\alpha \leqslant \nu$, and $\omega_\nu(f;t)_{L_p} = O(t^\alpha)$, then as $t \to 0$

$$\omega_k(f;t)_{L_p} = \begin{cases} O(t^k), & \text{if } \alpha > k, \\ O\left(t^k \ln \frac{1}{t}\right), & \text{if } \alpha = k, \\ O(t^\alpha), & \text{if } \alpha < k. \end{cases} \tag{6}$$

Moreover, replacing O by o (as $t \to 0$) in the assumption entails the same replacement in the last two relations of (6).

In particular, for $k = 1$, $\nu = 2$, $0 < \alpha \leqslant 1$, this yields the known result (^{1}), first discovered for periodic functions by Zygmund (^{2}).

There are examples showing that, as applied to the uniform metric ($p = \infty$), estimate (4) cannot in general be improved in order (as $t \to 0$). In these cases the orders (as $t \to 0$) of the left- and right-hand sides of (4) coincide. At the same time, in a number of cases estimate (4), while sharp in order (as $t \to 0$) for the uniform metric, turns out to be too crude for the metric L_p with $1 < p < \infty$.*

This circumstance could be illustrated quite simply by the example of the space L_2, when $\nu = k + 1$. In this case the inequality

$$\omega_k(f;t)_{L_2} \leqslant C_k t^k \left[\int_t^1 \frac{\omega_{k+1}^2(f;u)_{L_2}}{u^{2k+1}}\, du\right]^{1/2} \qquad (0 < t \leqslant 1/2).$$

is valid.

Corollary 2. If $k \geqslant 2$ and the function $f(x)$ on the whole real axis has a derivative $f'(x) \in L_p(-\infty, \infty)$ $(1 \leqslant p \leqslant \infty)$, then for $0 < t \leqslant 1/2$,

$$\frac{\omega_k(f;t)_{L_p}}{t^k} \leqslant C_k \int_t^1 \frac{\omega_k(f';u)_{L_p}}{u^k}\, du,$$

where C_k is a constant independent of f.

It would be interesting to determine the smallest value of the constant C_k in inequality (5). It is easy to see that, in the converse inequality (3) ($\nu = k+1$), the constant 2 in its right-hand side cannot be decreased.

In conclusion, we indicate two lemmas that may be used to prove inequality (4).

Lemma 1. If $A_\sigma(f)_{L_p}$ is the best approximation of the function $f(x)$ by entire functions of finite degree $\leqslant \sigma$ in the metric L_p on $(-\infty, \infty)$ $(1 \leqslant p \leqslant \infty)$, then for any $k \geqslant 1$

$$\omega_k\left(f; \frac{1}{n}\right)_{L_p} \leqslant \frac{B_k}{n^k} \sum_{i=1}^{n} i^{k-1} A_i(f)_{L_p}, \tag{7}$$

where B_k is a constant independent of the function f.

Lemma 2. For any function $f(x) \in L_p(-\infty, \infty)$ $(1 \leqslant p \leqslant \infty)$ the following inequality is valid:**

$$A_\sigma(f)_{L_p} \leqslant B_k \omega_k\left(f; \frac{1}{\sigma}\right)_{L_p}, \tag{8}$$

where B_k is a constant independent of the function f.

* In the periodic case, the indicated peculiarity of the metric L_p was noted by us earlier in ([4]). The result pertaining to this (see ([4]), Theorem 6) was subsequently supplemented by Zygmund ([3]).

** For the analogous inequality in the periodic case, see ([5]).

We note here that inequality (7) for periodic functions $f(x)$ was first indicated by us in ([4]) (see Theorem 6).

From Lemmas 1 and 2 there follow, as consequences, certain known direct and inverse theorems of the constructive theory of functions defined on the whole real axis ([1,3]). We also note that inequality (7), in the case when $1 < p < \infty$, admits an improvement. Thus, for example, when $p \geqslant 2$,

$$\omega_k\left(f; \frac{1}{n}\right)_{L_p} \leqslant \frac{B_{k,p}}{n^k} \left[\sum_{i=1}^{n} i^{2k-1} A_i^2(f)_{L_p}\right]^{1/2}, \tag{9}$$

and when $1 \leqslant p \leqslant 2$,

$$\omega_k\left(f; \frac{1}{n}\right)_{L_p} \leqslant \frac{B_{k,p}}{n^k} \left\{\sum_{\nu=1}^{n} \nu^{kp-1} A_\nu^p(f)_{L_p}\right\}^{1/p}.$$

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Note: Figure translations are in progress. See original paper for figures.