# ON CONJUGATE PROBLEMS IN THE THEORY OF THIN SHELLS

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**Abstract**

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**THEORY OF ELASTICITY**

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# ON CONJUGATE PROBLEMS IN THE THEORY OF THIN SHELLS

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Recently, in connection with the appearance of Hill' s work ([1]), interest has increased in G. V. Kolosov' s idea ([2]) concerning conjugate problems of the plane theory of elasticity. In the present note it is shown that the property of conjugacy can also be used in the theory of thin shells.

1. Suppose, for brevity of exposition, that the middle surface of the shell is bounded by a closed line of curvature $\alpha_1 = \text{const}$ (we adhere to the notation adopted in ([3])). In addition, we regard the problem as homogeneous and the Poisson coefficient $\mu$ as equal to zero. Then the generalized Hooke law and the relations introducing the stress functions $\bar{u}, \bar{v}, \bar{w}$ (the Lur' e–Goldenveizer functions) have the form

$$\begin{gathered} T_1 = Eh\varepsilon_1(u,v,w), \qquad T_2 = Eh\varepsilon_2, \qquad S = Eh \cdot {}^1/_2\omega, \\ M_1 = Ehc_0^2\chi_1, \qquad M_2 = Ehc_0^2\chi_2, \qquad H = Ehc_0^2\tau; \end{gathered} \tag{1}$$

$$\begin{gathered} T_1 = Ehc_0\bar{\chi}_2(\bar{u},\bar{v},\bar{w}), \qquad T_2 = Ehc_0\bar{\chi}_1, \qquad S = -Ehc_0\bar{\tau}, \\ M_1 = -Ehc_0\bar{\varepsilon}_2, \qquad M_2 = -Ehc_0\bar{\varepsilon}_1, \qquad H = Ehc_0 \cdot {}^1/_2\bar{\omega}, \end{gathered} \tag{2}$$

where $c_0 = h/\sqrt{12}$.

We introduce complex displacements and stresses by the relations

$$\tilde{u} = u + i\bar{u}, \qquad \tilde{v} = v + i\bar{v}, \qquad \tilde{w} = w + i\bar{w},$$

$$\tilde{T}_1 = T_1 - \frac{i}{c_0}M_2, \qquad \tilde{T}_2 = T_2 - \frac{i}{c_0}M_1, \qquad \tilde{S} = S + \frac{i}{c_0}H. \tag{3}$$

Forming complex combinations of the corresponding equations of systems (1) and (2), we obtain for the complex stresses the expressions

$$\tilde{T}_1 = T_1 - \frac{i}{c_0}M_2 = T_1 - iEhc_0\bar{\chi}_2 = -iEhc_0\tilde{\chi}_2(\tilde{u},\tilde{v},\tilde{w}),$$

$$\tilde{T}_2 = T_2 - \frac{i}{c_0} M_1 = T_2 - iEhc_0 \bar{\chi}_1 = -iEhc_0 \tilde{\chi}_1, \tag{4}$$

$$\tilde{S} = S + \frac{i}{c_0} H = S + iEhc_0 \tau = iEhc_0 \tilde{\tau}.$$

(Here and below the components of deformation marked with a bar depend on $\bar{u}, \bar{v}, \bar{w}$; those marked with a tilde–on $\tilde{u}, \tilde{v}, \tilde{w}$.)

Usually on the contour one prescribes either the generalized Kirchhoff forces–moments

$$T_1, \quad T'_{12} = T_{12} + \frac{M_{12}}{R_2}, \quad N'_1 = N_1 - \frac{1}{A_2} \frac{\partial M_{12}}{\partial \alpha_2}, \quad M_1;$$

or the generalized components of deformation

$$\varepsilon_2, \qquad \chi_2 = \frac{1}{A_2} \frac{\partial \psi}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \vartheta,$$

$$\chi'_{21} = \frac{1}{A_2} \frac{\partial \vartheta}{\partial \alpha_2} - \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} \psi - \frac{\omega_2}{R_2}, \qquad \chi'_{2n} = \frac{1}{A_2} \frac{\partial \omega_2}{\partial \alpha_2} + \frac{\vartheta}{R_2}, \tag{5}$$

where

$$-\psi = \frac{1}{A_2} \frac{\partial w}{\partial \alpha_2} - \frac{v}{R_2}, \qquad \vartheta, \ -\omega_2 = -\frac{1}{A_2} \frac{\partial u}{\partial \alpha_2} + \frac{1}{A_1 A_2} \frac{\partial A_2}{\partial \alpha_1} v$$

are the components of the rotation vector of the boundary element. The four quantities (5) completely determine the deformation of the boundary element.

In particular, the conditions of a clamped edge may be written in the form

$$\varepsilon_2 = \chi_2 = \chi'_{12} = \chi'_{2n} = 0. \tag{6}$$

In addition to the already introduced $\widetilde{T}_1 = T_1 - iEhc_0 \chi_2$, define on the contour the following complex quantities:

$$\widetilde{T}'_{21} = T'_{21} + iEhc_0 \chi'_{21}, \qquad \widetilde{N}'_1 = N'_1 - iEhc_0 \chi'_{2n}, \qquad \widetilde{M}_1 = M_1 + iEhc_0 \varepsilon_2.$$

Each of the written complex relations connects a pair of quantities–a static one and a geometric one. These quantities may appropriately be called conjugate.

Using relations (2), it is not difficult to obtain

$$\widetilde{T}_1 = T_1 - iEhc_0\chi_2 = -iEhc_0\tilde{\chi}_2, \qquad \widetilde{T}'_{12} = T'_{12} + iEhc_0\chi'_{21} = iEhc_0\tilde{\chi}'_{21},$$

$$\widetilde{N}'_1 = N'_1 - iEhc_0\chi'_{2n} = -iEhc_0\tilde{\chi}'_{2n}, \qquad \widetilde{M}_1 = M_1 + iEhc_0\varepsilon_2 = iEhc_0\tilde{\varepsilon}_2. \quad (7)$$

**2.** Consider, along with the basic deformed state, characterized by the complex displacements $\tilde{u}, \tilde{v}, \widetilde{w}$, the state (conjugate) for which

$$\tilde{u}^* = u^* + i\overline{u^*} = i\tilde{u}, \qquad \tilde{v}^* = v^* + i\overline{v^*} = i\tilde{v}, \qquad \widetilde{w}^* = w^* + i\overline{w^*} = i\widetilde{w}$$

$$(u^* = -\overline{u}, \quad \overline{u^*} = u, \quad v^* = -\overline{v}, \quad \overline{v^*} = v, \quad w^* = -\overline{w}, \quad \overline{w^*} = w). \qquad (8)$$

Substituting these expressions into (4) and separating the real and imaginary parts, we obtain, according to (1) and (2):

$$T_1^* = Ehc_0\chi_2 = \frac{1}{c_0}M_2, \qquad T_2^* = Ehc_0\chi_1 = \frac{1}{c_0}M_1, \qquad S^* = -Ehc_0\tau = -\frac{1}{c_0}H,$$

$$M_2^* = -Ehc_0^2\overline{\chi}_2 = -c_0T_1, \qquad M_1^* = -Ehc_0^2\overline{\chi}_1 = -c_0T_2, \qquad (9)$$

$$H^* = -Ehc_0^2\overline{\tau} = c_0S.$$

On the contour, according to (7) and (2):

$$T_1^* = Ehc_0\chi_2, \qquad T'^*_{12} = -Ehc_0\chi'_{21}, \qquad N'^*_1 = Ehc_0\chi'_{2n}, \qquad M_1^* = -Ehc_0\varepsilon_2,$$

$$\chi_2^* = -\overline{\chi}_2 = -\frac{1}{Ehc_0}T_1, \qquad \chi'^*_{21} = -\overline{\chi}'_{21} = \frac{1}{Ehc_0}T'_{12}, \qquad (10)$$

$$\chi'^*_{2n} = -\overline{\chi}'_{2n} = -\frac{1}{Ehc_0}N'_1, \qquad \varepsilon_2^* = -\overline{\varepsilon}_2 = \frac{1}{Ehc_0}M_1.$$

Consider two types of boundary conditions.

1) On the contour, four statico-geometric quantities are prescribed (one from each pair of conjugates). In this case, for the conjugate problem the boundary conditions pass into conjugate ones. Thus, if in the basic problem on the contour only geometric quantities are prescribed,

$$\varepsilon_2 = \varepsilon_2^0, \qquad \chi_2 = \chi_2^0, \qquad \chi'_{21} = \chi'^0_{21}, \qquad \chi'_{2n} = \chi'^0_{2n},$$

then in the adjoint problem the boundary conditions will be purely static:

$$T_1^* = Ehc_0\chi_2^0, \qquad T'^*_{12} = -Ehc_0\chi'^0_{21}, \qquad N'^*_1 = Ehc_0\chi'^0_{2n}, \qquad M_1^* = -Ehc_0\varepsilon_2^0$$

and conversely.

In particular, the conditions of a clamped edge (6) pass into the conditions of a free edge

$$T_1^* = T'^*_{12} = N'^*_1 = M_1^* = 0.$$

2) In the fundamental problem, the displacements $u = u^0$, $v = v^0$, $w = w^0$, $\vartheta = \vartheta_0$ are prescribed on the contour. According to (5), these completely determine the quantities $\varepsilon_2$, $\chi_2$, $\chi'_{21}$, $\chi'_{2n}$. Thus, for the adjoint problem we shall again have purely static boundary conditions. In particular, the clamping conditions $u = v = w = \vartheta = 0$ pass into the conditions of a free edge.

The noted property, closely connected with the static-geometric analogy, makes it possible, from the solution found for the fundamental problem, immediately to write down the solution of the adjoint problem.

If the boundary conditions are formulated in complex form, then the boundary conditions of the adjoint problem will be of the same type. In this sense such a problem may be called self-adjoint. It is not very difficult to remove the simplifying restrictions stated at the beginning of the article.

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## CITED LITERATURE

1. R. Hill, *Collection of Abridged Translations*, Mechanics, No. 6, 1956, p. 71.

2. G. V. Kolosov, *Application of a Complex Variable to the Theory of Elasticity*, L.–M., 1935, p. 71.

3. V. V. Novozhilov, *Theory of Thin Shells*, 1951.