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Abstract

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ON THE QUESTION OF THE PULSATION OF A PLASMA PINCH

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The problem of the pulsation of a plasma pinch was posed in connection with experimental studies of a spark of very large current strength in a gas ([1]). An approximate solution of this problem was given by M. A. Leontovich and S. M. Osovets ([2]). Here we propose a certain class of exact solutions of the equations of magnetohydrodynamics, containing, in particular, periodic solutions.

Let us consider one-dimensional axisymmetric motions of an unbounded gas with infinite conductivity. The well-known equations of magnetohydrodynamics in Lagrangian coordinates are written in the following form:

$$\frac{\partial^2 r}{\partial t^2} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r} + \frac{1}{8\pi}\frac{\partial H^2}{\partial r} + \frac{1}{4\pi}\frac{H^2}{r}\right);$$

$$\rho = \rho_0 \frac{r_0}{r}\frac{\partial r_0}{\partial r}; \qquad p = p_0\frac{\rho^\gamma}{\rho_0^\gamma}; \qquad H = H_0\frac{\partial r_0}{\partial r}. \tag{1}$$

Here r is the current coordinate of the particle; r_0, ρ_0, p_0, H_0 are, respectively, the coordinate, density, pressure, and magnetic intensity of the particle at $t = 0$, while the magnetic lines of force are assumed to be closed concentric circles.

The first equation expresses the acceleration of the particle; the second, third, and fourth represent the laws of conservation of mass, entropy, and magnetic flux. In this work all solutions with homogeneous deformation are found, i.e., such that

$$\frac{r}{r_0} = \mu(t), \tag{2}$$

where μ does not depend on r_0. In this case

$$v = \frac{\partial r}{\partial t} = r_0\mu'(t) = r\frac{\mu'(t)}{\mu(t)}, \tag{3}$$

i.e., the velocity depends linearly on the radius. These solutions are analogous to the solution obtained by L. I. Sedov ([3], [4]) in the absence of magnetic forces, which contains, as does the solution proposed here, one arbitrary function.

The solution is given by the formulas:

$$\rho = \rho_0 \mu^{-2}; \tag{4}$$

$$p = p_0 \mu^{-2\gamma}; \tag{5}$$

$$H^2 = H_0^2 \mu^{-2}, \tag{6}$$

where ρ_0 is an arbitrary function of r_0;

$$p_0 = A \int_0^{r_0} \rho_0 r_0 \, dr_0 + N; \tag{7}$$

$$H_0^2 = \frac{8\pi B}{r_0^2} \int_0^{r_0} \rho_0 r_0^3 \, dr_0 + \frac{M}{r_0^2}. \tag{8}$$

Here A, B, N, and M are arbitrary constants. Setting $r_0 = 0$, we obtain that $N \geq 0$ and $M \geq 0$. The sign of A coincides with the sign of dp_0/dr_0. The sign of B coincides with the sign of the first term in formula (8). From Maxwell' s equation

$$\frac{4\pi}{c} j = \operatorname{rot} H \tag{9}$$

we find that at $t = 0$ a concentrated current

$$i_0 = \pm \frac{c}{2} \sqrt{M}$$

flows along the axis of symmetry and, in addition, at all other points of space there is a distributed current with density

$$j_0 = \pm \frac{c}{8\pi} \frac{B\varphi'(r_0)}{\sqrt{M + B\varphi(r_0)}}, \qquad \text{where } \varphi(r_0) = 8\pi \int_0^{r_0} \rho_0 r_0^3 \, dr_0.$$

Thus, if $B > 0$, the distributed current coincides in direction with the concentrated current along the axis of symmetry; if $B < 0$, these currents are opposite. If, however, $M = 0$, i.e., there is no concentrated current, then B must necessarily be positive.

For $\mu(t)$ one obtains the differential equation

$$\left(\frac{d\mu}{dt}\right)^2 = \frac{A}{\gamma - 1}\mu^{2-2\gamma} - 2B\ln\mu + C = f(\mu) \tag{10}$$

(C is an arbitrary constant), whence

$$t = \pm\int \frac{d\mu}{\sqrt{f(\mu)}}.$$

The following cases are possible:

1) $A > 0$, $B > 0$. If $d\mu/dt < 0$ at $t = 0$, then μ becomes equal to zero after a finite time: collapse occurs, the gas is compressed into a point. If, however, $d\mu/dt > 0$, then μ first reaches its maximum value, and then exactly the same collapse occurs. One may assume that at the moment of collapse the velocity changes discontinuously and expansion again occurs.

2) $A < 0$, $B < 0$. If $d\mu/dt > 0$ at $t = 0$, then expansion occurs: μ increases to infinity as $t \to \infty$. In this case the velocity $d\mu/dt \to \infty$. If, however, $d\mu/dt < 0$ at $t = 0$, then μ first reaches its minimum value, and thereafter expansion occurs.

3) $A > 0$, $B < 0$. The following subcases are possible:

a) $f(\mu)$ has no roots. Then, if $d\mu/dt > 0$ at $t = 0$, expansion occurs, and if $d\mu/dt < 0$ at $t = 0$, collapse occurs.

b) $f(\mu)$ has one double root μ_0. Then limiting motions $\mu \to \mu_0$ as $t \to \infty$ are possible. Otherwise there will be expansion or collapse, depending on the sign of $\mu_0 - 1$. If $\mu_0 = 1$, then this is a position of unstable equilibrium.

c) $f(\mu)$ has two roots, and 1 cannot lie between them. If $1 \leqslant \mu_1 < \mu_2$, then the motion is qualitatively the same as in case 1). If, however, $\mu_1 < \mu_2 \leqslant 1$, then the motion proceeds analogously to case 2).

4) $A < 0$, $B > 0$. In this case there exist either two roots $\mu_1 < 1 < \mu_2$, or one double root $\mu_1 = \mu_2 = 1$. In the case of the double root, $\mu = 1$ is a position of stable equilibrium. For $\mu_1 < 1 < \mu_2$, periodic oscillations occur with a period depending on the constants A, B, and C.

5) $A = 0$—the pressure is constant. If $B > 0$, the motion proceeds analogously to case 1). If $B < 0$, the motion proceeds analogously to case 2).

6) $B = 0$—the magnetic forces are in equilibrium. Let $A > 0$. If $f(\mu)$ has a root, then the motion proceeds analogously to case 1). If there is no root, then for $d\mu/dt > 0$ expansion occurs with velocity $d\mu/dt \to 1/\sqrt{C}$, while for $d\mu/dt < 0$ collapse occurs. If $A < 0$, then the motion proceeds analogously to case 2), but $d\mu/dt \to 1/\sqrt{C}$ as $t \to \infty$.

Let us now consider a cylinder of finite length and finite radius $r_0 = R$. In order that the phenomenon inside the cylinder be described by the solution obtained, it is sufficient to place conducting walls at the ends, and to apply to its lateral surface a pressure equal to the total pressure from within,

$$p^*(R,t) = p_0(R)\mu^{-2\gamma} + \frac{1}{8\pi} H_0^2(R)\mu^{-2}.$$

This pressure may be produced by a piston, but it may also be produced by an external electromagnetic field. The pressure of the electromagnetic field on the surface of the cylinder may be taken equal to $\frac{1}{8\pi}H_v^2$ (H_v is the intensity of the external field). According to equation (9), the total current flowing along the cylinder must then be equal to

$$I(t) = \frac{c}{2} R\mu H_v = \frac{c}{2} R\sqrt{8\pi p_0(R)\mu^{2-2\gamma} + H_0^2(R)}.$$

In particular, when $p_0(R) = 0$, the total current is constant in time.

Thus, in order that the phenomenon inside the cylinder be described by the solution obtained, it is necessary to apply to the ends of the cylinder such a voltage that the total current be equal to $I(t)$. Such a voltage can always be applied, and it can readily be calculated. In this case the current flowing inside the cylinder does not depend on the applied voltage, but is determined from the equations of magnetic hydrodynamics. The applied voltage affects only the magnitude of the surface current.

Let us now consider the energy balance. The work of the external forces

$$2\pi R^2 p^*(\mu)\,\mu\,d\mu = 2\pi R^2 \left(\mu^{1-2\gamma} A \int_0^R \rho_0 r_0\,dr_0 + \mu^{1-2\gamma} N + \mu^{-1}\frac{B}{R^2}\int_0^R \rho_0 r_0^3\,dr_0 + \mu^{-1}\frac{8\pi M}{R^2} \right) d\mu \tag{11}$$

is expended on increasing the internal energy and on creating kinetic energy. The last term in this formula does not depend on R, i.e., energy is pumped toward the axis of symmetry, where for $M \neq 0$ the energy is infinite. The first three terms tend to zero as $R \to 0$, hence–

Consequently, the energy corresponding to these terms and flowing in through the boundary is distributed over the entire mass of the fluid. It is seen from formula (6) that the magnetic energy contained in the particle remains constant. Therefore one may consider that the third term goes entirely into increasing the kinetic energy, the second entirely into increasing the internal energy of the gas, while the first goes partly into increasing the kinetic energy and partly into increasing the internal energy of the gas. During collapse the external forces perform infinite work on the cylinder, and each term of formula (11) is infinite.

During expansion the magnetic forces (the last two terms) likewise perform infinite work, which is done at the expense of the infinite energy at the center. In the case of oscillations, the work performed by the surface of the cylinder is finite.

The solution obtained in this paper is readily generalized to the case where the magnetic lines of force are helical lines. In this case, to the right-hand side of equality (1) there is added the term $-\frac{1}{8\pi\rho}\frac{\partial H_z^2}{\partial r}$, where H_z is the component of the magnetic field parallel to the axis of symmetry; H in this equation will denote the component of the magnetic field perpendicular to the axis of symmetry. Then to formulas (4), (5), (6), (7), (8) there are added the formulas

$$H_z^2 = H_{z0}^2 \mu^{-4}, \tag{6'}$$

$$H_{z0} = 8\pi D \int_0^{r_0} \rho_0 r_0' dr_0' + L \tag{8'}$$

(D and L are arbitrary constants), and instead of equation (10) we obtain:

$$\left(\frac{\partial \mu}{\partial t}\right)^2 = \frac{A}{\gamma - 1}\mu^{2-2\gamma} - 2B\ln\mu - D\mu^{-2} + C. \tag{10'}$$

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REFERENCES

[1] L. A. Artsimovich, A. M. Andrianov, O. A. Bazilevskaya, Yu. G. Prokhorov, N. V. Filippov, *Atomic Energy*, No. 3 (1956). [2] M. A. Leontovich, S. M. Osovets, *Atomic Energy*, No. 3 (1956). [3] L. I. Sedov, DAN, 90, No. 5, 735 (1953). [4] L. I. Sedov, *Methods of Dimension and Similarity in Mechanics*, Moscow, 1954, p. 237.

Note: Figure translations are in progress. See original paper for figures.