ELASTICITY THEORY

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Abstract

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ON CERTAIN INTEGRAL CHARACTERISTICS OF THE STRESS STATE IN THE BENDING OF PRISMATIC BARS

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The paper gives a rather simple method for obtaining estimates for the maximum stresses in the case of bending of prismatic bars, based on the use of special properties of subharmonic functions,* and establishes certain general properties of the stress state that characterize it as a whole.

The shear stresses in a prismatic bar (a cantilever) with cross-section G, under bending by its transverse force P, directed along the axis of inertia coinciding with the x-axis and applied at the center of gravity of the end cross-section, are known to be determined by the equalities (see, for example, (^{2}))

$$X_z = \mu\tau\left(\frac{\partial\varphi}{\partial x} - y\right) - \frac{P}{2(1-\sigma)I}\left[\frac{\partial\chi}{\partial x} + \frac{\sigma x^2}{2} + \left(1 - \frac{\sigma}{2}\right)y^2\right],$$

$$Y_z = \mu\tau\left(\frac{\partial\varphi}{\partial y} + x\right) - \frac{P}{2(1+\sigma)I}\left[\frac{\partial\chi}{\partial y} + (2+\sigma)xy\right], \qquad (1)$$

where $\varphi(x, y)$ is the torsion function; $\chi(x, y)$ is the bending function; I is the moment of inertia of the domain G with respect to the y-axis; the origin of coordinates coincides with the center of gravity of the domain G; μ is the shear modulus; σ is Poisson' s ratio; τ is a constant called the twist.

Let us consider especially the case in which the domain G is symmetric with respect to the x-axis. Using the principle of symmetry under analytic continuation, one can show that in this case the bending function will be an even function of y, and the constant τ is equal to zero. If, moreover, one introduces the "complex bending function" $W(z) = U + iV$ $(z = x + iy)$ by means of the equalities

$$U(x, y) = -\chi(x, y) - \left(1 + \frac{\sigma}{2}\right)x\left(y^2 - \frac{x^3}{3}\right),$$

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x},$$

then, as direct calculations show, the shear-stress vector $\mathbf{V} = X_z + iY_z$ will be determined by the equality

$$\mathbf{V} = \frac{P}{2(1+\sigma)I}\left\{\overline{\frac{\partial w}{\partial z}} - [(1+\sigma)x^2 - \sigma y^2]\right\}, \tag{2}$$

where the complex bending function $w = U + iV$ is analytic in the domain G and is determined by the boundary conditions

$$V\big|_{L_k} = -\frac{\sigma y^3}{3} + (1+\sigma)\int_{L_k} x^2\, dy + c_k, \tag{3}$$

* The definition and properties of subharmonic functions are given in the monograph of I. I. Privalov (1).

where c_k are constants on each of the contours L_k that form part of the boundary L of the domain G.

It can be shown that, for an increment along some contour Γ lying inside the domain G, the real part of the complex bending function $[U]_\Gamma$ and the imaginary part of this function $[V]_\Gamma$ are given by the following simple formulas:

$$[U]_\Gamma = \frac{2(1+\sigma)I}{P}\int_\Gamma V_s\, ds + \int_\Gamma [(1+\sigma)x^2 - \sigma y^2]\, dx, \tag{4}$$

$$[V]_\Gamma = \frac{2(1+\sigma)I}{P}\int_\Gamma V_n\, ds + \int_\Gamma [(1+\sigma)x^2 - \sigma y^2]\, dy, \tag{5}$$

where V_s is the projection of the vector of tangential stresses $\mathbf{V}$ onto the positive direction of the tangent to the contour Γ; V_n is the projection of this vector onto the right normal to this contour.

From formulas (2), (4), and (5) one immediately obtains a number of corollaries which, for the case of the rods under consideration, characterize the stress state as a whole.

1. The stress state is symmetric with respect to the x-axis.
2. If the closed contour Γ bounds a simply connected domain T lying inside the domain G, then the equalities

$$\int_\Gamma V_n\, ds = -\frac{PS^* x^*}{I}, \qquad \int_\Gamma V_s\, ds = -\frac{PS^* y^*}{(1+\sigma)I} \qquad \left(S^*(x^* + iy^*) = \iint_T z\, dx\, dy\right)^* \tag{6}$$

hold.

3. In particular, if the domain T is a circle of radius r, then

$$\int_\Gamma V_n\,ds = -\frac{P\pi r^2 x^*}{I}, \qquad r^2\int_\Gamma V_s\,ds = -\frac{P\pi r^4 \sigma y^*}{(1+\sigma)I}. \tag{7}$$

The left-hand side of the last of these equalities is the moment of the forces applied to the boundary of the circle T with respect to its center; the left-hand side of the first of these equalities may be regarded as the "total tensile force," characterizing the forces applied to the boundary of the circle T and acting in tension.

4. Let Γ be a curve (or a collection of curves) dividing the domain G into two domains; let G^* be that one of the latter domains which lies to the left of Γ and is simply connected. Then the equality

$$\int_\Gamma V_n\,ds = -\frac{PS^*x^*}{I} \qquad \left(S^*x^* = \iint_{G^*} x\,dx\,dy\right) \tag{8}$$

will hold.

5. Let Γ be a curve without multiple points, the ends of this curve lying on the boundary of the domain G and being located symmetrically with respect to the x-axis; then the equality

$$\int_\Gamma V_s\,ds = -\frac{P}{2(1+\sigma)I}\int_\Gamma [(1+\sigma)x^2 - \sigma y^2]\,dy \tag{9}$$

will hold.

6. From formulas (8) and (9), as a special case, the following simple theorem is obtained.

Theorem 1. *If Γ is a rectilinear segment perpendicular to the x-axis (or a collection of such segments), dividing the domain G into two domains; G^* is that one of the latter domains which lies to the left of Γ and is simply connected, then for the components X and Y of the resultant of the forces applied to the segment Γ in the directions of the x- and y-axes, the equalities*

$$X = -\frac{PS^*x^*}{I}, \qquad Y = 0, \qquad \left(S^*x^* = \iint_{G^*} x\,dx\,dy\right) \tag{10}$$

hold.

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[*] The second of these formulas, as applied to the case under consideration, coincides with the formula for the circulation of the vector of tangential stresses established by L. S. Leibenson [3].

Fig. 1

Figure 1: Fig. 1

We note that if one introduces an additional simplifying hypothesis, namely, if one assumes approximately that the normal component of the forces along the segment Γ remains constant, then from formulas (10) one obtains the well-known Zhuravskii formula for shear stresses

$$X_z = -\frac{PS_x^*}{Ib(x)}, \tag{11}$$

where $b(x)$ is the length of the above-mentioned rectilinear segment Γ (4).

It follows from formula (2) that estimating the magnitude of the stress vector in the domain G reduces to the solution of the following special problem in the theory of functions of a complex variable.

It is required to estimate the magnitude of the derivative of the function $w = U + iV$, analytic in the domain G, symmetric with respect to the x-axis, if the imaginary part of this function on the boundary L of this domain is determined by the boundary condition (3), where the origin is assumed to coincide with the centroid of the domain G.

Fig. 1

It should be noted that special case of this problem in which the domain G is simply connected, or all contours L_k forming its boundary L are symmetric with respect to the x-axis. In this case it can be shown that in the boundary condition (3) all constants c_k may be taken equal to zero, assuming that the function $V(x, y)$ is equal to zero at every point x_k which is the point of intersection of the contour L_k with the positive direction of the x-axis.

For the solution of the auxiliary problem in the theory of functions just posed, the following theorem* proves very useful.

Theorem 2 (on majorization of the derivative). Let $u(x, y)$ be a function harmonic in the domain D, having a piecewise smooth boundary C, and continuously differentiable in the closed domain $D + C$; C_1 is a part of C; $u_1(x, y)$, $u_2(x, y)$ are arbitrary functions, twice continuously differentiable in the domain D and continuously differentiable in the closed domain $D + C$. Then, if the function $u_1(x, y)$ is superharmonic and on the boundary of the domain satisfies the conditions

$$u_1\big|_{C_1} = u\big|_{C_1}, \qquad u_1\big|_{C-C_1} \geq u\big|_{C-C_1}, \tag{12}$$

and the function $u_2(x, y)$ is subharmonic and satisfies the conditions

$$u_2\big|_{C_1} = u\big|_{C_1}, \qquad u_2\big|_{C-C_1} \leq u\big|_{C-C_1}, \tag{13}$$

then on C_1 the inequality holds

$$|\nabla u| \leq \max\{|\nabla u_1|,\ |\nabla u_2|\}, \tag{14}$$

where

$$\nabla u = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}.$$

The proof of this theorem is obtained by considering the properties of subharmonic functions.

With the aid of Theorem 2, it becomes possible, using the simplest auxiliary functions $u_1(x, y)$ and $u_2(x, y)$, to obtain estimates of the magnitude of the stress vector in many cases when the exact solution of the problem of bending of a prismatic bar is either very complicated or cannot be found in explicit form at all.

For example, for the case of the cross section of the bar shown in Fig. 1, where $\alpha\beta$ is an arc of a semicircle of radius $r < \min(a, b) = c$ with center at the point $(0, b)$; $\gamma\delta$ is an arc symmetric to the arc $\alpha\beta$ with respect to the x-axis, i.e. for a bar with a cross section close to an I-section, an estimate of the ma— of the magnitude of the stress vector over the entire closed domain of the cross section can be obtained in the form

$$|\mathbf{V}| \leq \frac{P}{2(1+\sigma)I}\left[\sigma b^2 + (1+\sigma)a^2 + (1+\sigma)\max\{Q_1, Q_2, Q_3\} + \left|(1+\sigma)x^2 - \sigma y^2\right|\right], \tag{15}$$

where

$$Q_1 = a\sqrt{a^2 + 4b^2}, \qquad Q_2 = \frac{R^2 b}{b - r}, \tag{16}$$

$$Q_3 = b^2 + r^2 - a^2 + bR^3 \max\left\{\frac{2}{r^2}, \frac{c}{(c-r)r^2}\right\}, \qquad R = \sqrt{a^2 + b^2} \geqslant 1.$$

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* A problem of this kind, i.e. the problem of estimating the derivative of an analytic function in an arbitrarily prescribed domain from the values of its imaginary part on the boundary of the domain, has not, so far as we know, been studied specially in the theory of functions of a complex variable.

In particular, for $r = 0$, i.e., for the case of a rectangular cross section, for which, in contrast to the case $r \neq 0$, the exact solution of the problem is known, these estimates can be obtained by the method indicated above in the form

$$|\mathbf{V}| \leq \begin{cases} \dfrac{P}{2(1+\sigma)I}\left[(1+\sigma)a^2 + \left|(1+\sigma)x^2 - \sigma y^2\right|\right], & \text{if } b \leq a\sqrt{2\left(\dfrac{1}{\sigma}-1\right)}, \\ \dfrac{P}{2(1+\sigma)I}\left[\sqrt{(1+\sigma)a^4 - 2\sigma(1-\sigma)a^2b^2 + \sigma^2 b^4} + \left|(1+\sigma)x^2 - \sigma y^2\right|\right], & \text{if } b \geq a\sqrt{2\left(\dfrac{1}{\sigma}-1\right)}. \end{cases} \tag{17}$$

The latter estimates, as comparison with the exact solution known for this case shows, are close to the exact values of the magnitude of the stress vector (5).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.