MATHEMATICS

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Abstract

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ON THE LOCAL INFINITE DIVISIBILITY OF MARKOV PROCESSES

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§ 1. As is known, Markov processes homogeneous in space and time are infinitely divisible, i.e., the logarithm of the characteristic function of the transition probability of such a process is represented by the Lévy-Khintchine formula. Here a modification of this fact is considered in application to stochastic processes of a general type, including Markov processes inhomogeneous in space and time.

The following definition will be basic for what follows:

Definition 1. We shall say that a family of distribution functions $\{P(t,x)\}$, $t \in T$, where T is an arbitrary point set of the real line, is **right differentiable** at the point t_0, if

$$\lim_{\substack{t \to t_0+0 \\ t \in T}} \frac{1}{t-t_0} \int_{-\infty}^{+\infty} (e^{izx}-1)\, d_{xP}(t,x) = \varphi_{t_0}^{+}(z), \tag{1}$$

where $\varphi_{t_0}^{+}(z)$ is a function finite on $(-\infty,+\infty)$, and the limit is uniform in z on every finite interval. If this limit exists uniformly in z on every finite interval as $t \to t_0 - 0$ and $t \in T$, then the family $\{P(t,x)\}$ is called **left differentiable** at the point t_0.

For families differentiable in the sense of this definition the following proposition is valid.

Theorem 1. If the family of distribution functions $\{P(t,x)\}$ with parameter $t \in T$ is right differentiable at the point t_0, then $\varphi_{t_0}^{+}(z)$ is represented by the Lévy-Khintchine formula, i.e.,

$$\varphi_{t_0}^{+}(z) = izm_{t_0}^{+} + \int_{-\infty}^{+\infty} \left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) \frac{1+x^2}{x^2}\, d_x G_{t_0}^{+}(x), \tag{2}$$

where $G_{t_0}^{+}(x)$ is a bounded nondecreasing function. The function $G_{t_0}^{+}(x)$ is uniquely determined at its points of continuity. An analogous proposition holds in the case of left differentiability.

Conditions ensuring the differentiability of a family in the sense of Definition 1 are given by the following theorem.

Theorem 2. For the right differentiability of the family $\{P(t,x)\}$ with parameter $t \in T$ at the point t_0, it is necessary and sufficient that the following conditions be fulfilled:

1) There exists the finite limit

$$\lim_{\substack{t\to t_0+0\\ t\in T}} \frac{P(t,x)-E(x)}{t-t_0} = \begin{cases} Q_1(x), & \text{for } x<0,\\ Q_2(x), & \text{for } x>0, \end{cases} \tag{3}$$

for each x belonging to some set L dense in $(-\infty,+\infty)$. Here and below $E(x) = 1$ for $x \geq 0$; $E(x) = 0$ for $x < 0$. The functions $Q_1(x)$, $Q_2(x)$ are monotonically nondecreasing respectively on $(-\infty, 0)$ and $(0, +\infty)$ and satisfy the condition $Q_1(x) \to 0$ as $x \to -\infty$ and $Q_2(x) \to 0$ as $x \to +\infty$.

2) For every pair of points $a \in L,\ b \in L$ the limiting relations hold, for $k = 1, 2,$

$$\lim_{\substack{t\to t_0+0\\ t\in T}} \frac{1}{t-t_0} \int_a^b x^k\, d_{xP}(t,x) = C_k(a,b), \qquad |C_k(a,b)| < \infty. \tag{4}$$

An analogous assertion is also valid for the case when the family $\{P(t,x)\}$, $t \in T$, is differentiable from the left at the point t_0.

Remark. Taking condition 1) into account, it is easy to show that in condition 2) it is sufficient to require the fulfillment of the limiting relations (4) only for one pair of points a and b such that $a \in L,\ b \in L$ and $a < 0,\ b > 0$.

The formulated Theorems 1 and 2, obviously, are easily reformulated for arbitrary Markov processes, homogeneous and nonhomogeneous both in space and in time.

The following theorem clarifies what smoothness requirements, imposed on the transition distribution functions of a Markov process, can ensure differentiability of this process in the case of a process homogeneous in time.

Theorem 3. Let a Markov process homogeneous in time, specified by transition distribution functions $P(t,x,y)$, satisfy the conditions:

1) Condition 1) of Theorem 2 is fulfilled for the family $\{P(t,x_0,x_0+y)\}$, $t \geq 0$, for some fixed $x_0 \in (-\infty,+\infty)$ and $t \geq 0$.

2) For fixed $y_0 \in (-\infty,+\infty)$ and all $t,\ t_2 \geq t \geq t_1 \geq 0$, the function $P(t,x_0,y_0)$ is continuous in t, and $P(t,x,y_0)$ is continuous in $x \in (-\infty,+\infty)$ for every fixed $t \in [t_1,t_2]$.

3) The transition distribution function $P(t, x, y)$, as a function of the variable x, has in $[x_0 - \alpha,\ x_0 + \beta]$ derivatives up to the second order inclusive, $P'_x(t, x, y_0)$ and $P''_{xx}(t, x, y_0)$, for the aforementioned value y_0, and these derivatives are continuous in x in $[x_0 - \alpha,\ x_0 + \beta]$, this continuity being uniform with respect to $t \in [t_1, t_2]$. The functions $P'_x(t, x_0, y_0)$ and $P''_{xx}(t, x_0, y_0)$ are continuous in t in $[t_1, t_2]$.

4) In the interval (t_1, t_2) there exist two values $t = s_1$ and $t = s_2$ for which

$$\begin{vmatrix} P'_x(s_1, x_0, y_0) & P'_x(s_2, x_0, y_0) \\ P''_{xx}(s_1, x_0, y_0) & P''_{xx}(s_2, x_0, y_0) \end{vmatrix} \neq 0.$$

Then the Markov process is differentiable from the right at the point $t = 0$ for $x = x_0$.

In the case of Markov processes homogeneous in space and in time, from infinite divisibility at the point $t_0 = 0$, as is known, there follows infinite divisibility of the process for any $t_0 > 0$. In the case considered here of processes nonhomogeneous in space, one obtains a certain modification of this fact. For processes homogeneous in time, for example, the following theorem is valid.

Theorem 4. If a Markov process $P(t, x, y)$, $t \geq 0$, homogeneous in time, is differentiable from the right at the point $t = 0$ for every $x \in (-\infty, +\infty)$, and if the inequality

$$\left| \frac{1}{t} \int_{-\infty}^{+\infty} (e^{izy} - 1)\, d_{yP}(t, x, x + y) \right| \leq M(a, b)$$

is satisfied simultaneously for all $x \in (-\infty, +\infty)$ and $z \in [a, b]$, $-\infty < a < b < +\infty$, and all t sufficiently close to 0, $0 < t < t < t'$, then for any $t_0 > 0$ the formula holds:

$$\frac{\partial^+}{\partial t} \int_{-\infty}^{+\infty} e^{izy}\, d_{yP}(t, x, y) \bigg|_{t=t_0} =$$

$$= \int_{-\infty}^{+\infty} e^{iz\xi} \left[izm(\xi) + \int_{-\infty}^{+\infty} \left(e^{izy} - 1 - \frac{izy}{1 + y^2} \right) \frac{1 + y^2}{y^2}\, d_{yG}(\xi, y) \right] d_\xi P(t, x, \xi), \tag{5}$$

where $m(x)$ and $G(x, y)$ are the functions determined by Theorem 1 as applied to the Markov process. If, moreover, the right-hand side of equality (5) is continuous in $t_0 > 0$, then on the left there will stand the two-sided derivative.

For processes nonhomogeneous in time, analogous theorems are valid, but with account taken of right and left differentiability.

Theorem 4 is a generalization of the Lévy-Khintchine formula to processes nonhomogeneous in space, since in the case of a process homogeneous in space the Lévy-Khintchine formula is obtained immediately from equation (5).

§ 2. We shall indicate, for Markov processes homogeneous in time, the solution of the inverse problem, which we formulate in the following form. A function of the form is given

$$\alpha(x,z) = izm(x) + \int_{-\infty}^{+\infty} \left(e^{iz\xi} - 1 - \frac{iz\xi}{1+\xi^2}\right) \frac{1+\xi^2}{\xi^2}\, d_\xi G(x,\xi). \tag{6}$$

What restrictions must be imposed on this function, or, what is the same, on the functions $m(x)$ and $G(x,y)$, in order that there exist a Markov process homogeneous in time with transition distribution function $P(t,x,y)$, differentiable from the right at the point $t=0$ for every $x \in (-\infty,+\infty)$?

Remark. The solution of the problem considered here is given in [1] for the case of nonhomogeneous processes, but under stronger, in a certain sense, restrictions on the function $\alpha(x,z)$ than those which will be formulated here. We note that Ito defines differentiability of a process from somewhat different positions, but our definitions and his are close to one another.

The following two theorems give the solution of the indicated problem.

Theorem 5. Let the following conditions be satisfied for the function $\alpha(x,z)$ represented by formula (6):

1) The function $m(x)$ is continuous and bounded on the entire line $(-\infty,+\infty)$.
2) The function $G(x,y)$, for every fixed $x \in (-\infty,+\infty)$, is a nondecreasing function of bounded variation in y on $(-\infty,+\infty)$.
3) As a function of the variable x, $G(x,y)$ is weakly continuous on $(-\infty,+\infty)$, i.e.

$$\int_{-\infty}^{+\infty} f(y)\, d_{yG}(x,y)$$

is a continuous function of x on $(-\infty,+\infty)$ for every continuous $f(y)$ on $(-\infty,+\infty)$.

4) For all values $x \in (-\infty,+\infty)$,

$$\operatorname*{Var}_{-\infty<y<+\infty} G(x,y) \le c < \infty$$

and the conditions $G(x,y) \to G(x,-\infty)$ and $G(x,y) \to G(x,+\infty)$ as $y \to \mp\infty$ are fulfilled uniformly in all $x \in (-\infty,+\infty)$.

5) The integral equation

$$\int_{-\infty}^{+\infty} e^{izx}\alpha(x,z)\,d\Phi(x) - \lambda\int_{-\infty}^{+\infty} e^{izx}\,d\Phi(x) = 0$$

for at least one $\lambda > 0$ has no nontrivial solutions, $\Phi(x) \not\equiv \text{const}$, among functions of bounded variation $\Phi(x)$ on $(-\infty, +\infty)$ such that

$$\int_{-\infty}^{+\infty} d\Phi(x) = 0.$$

Then there exists a time-homogeneous Markov process, specified by the transition distribution function $P(t,x,y)$, and satisfying, for every $x \in (-\infty, +\infty)$, the condition

$$\lim_{t\to 0+} \frac{1}{t}\int_{-\infty}^{+\infty} (e^{izy} - 1)\,d_y P(t, x, x+y) = \alpha(x,z).$$

The following theorem makes it possible to replace condition 5) of Theorem 5 by a stronger, but at the same time simpler, requirement.

Theorem 6. *Theorem 5 remains valid if, instead of condition 5), one requires that, as $x \to \pm\infty$, $\alpha(x,z) \to 0$ uniformly in z on every finite interval.*

The following theorem establishes the uniqueness of the processes determined by Theorems 5 and 6.

Theorem 7. *In the class of time-homogeneous Markov processes $P(t,x,y)$, measurable in x and in t in the Lebesgue sense, there exists a unique process, right-differentiable at the point $t = 0$ for every $x \in (-\infty, +\infty)$, and such that $\alpha(x,z)$ of the form (6) satisfies all the requirements of Theorem 5 or 6.*

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1. Ito Kiyosi, *Mem. Am. Math. Soc.*, No. 4 (1951).

Note: Figure translations are in progress. See original paper for figures.