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Abstract

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THEORY OF ELASTICITY

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ON THE EXISTENCE OF SOLUTIONS IN THE NONLINEAR THEORY OF SHELLS

(Presented by Academician S. L. Sobolev on 31 V 1957)

1. In the works ^(1,2) the existence of solutions was shown for the problem of large deflections of a shell whose middle surface differs little from a plane, and in the presence only of a transverse load. In the present note the general case is considered. In addition, a new method of proving the existence theorem will be applied here, making it possible at the same time to establish certain general properties of the solutions.
2. Let the middle surface of the shell S be given by the equation $\mathbf{r} = \mathbf{r}(\alpha, \beta)$, and suppose that the following conditions are satisfied:
 - 1) $\alpha, \beta \in \bar{\Omega}$, where Ω is some bounded domain of the α, β -plane;
 - 2) the boundary of Ω , Γ_{Ω} , consists of a finite number of arcs, on each of which the tangent rotates continuously;
 - 3) S has an isothermic net α_1, β_1 , with $\alpha = \alpha(\alpha_1, \beta_1)$, $\beta = \beta(\alpha_1, \beta_1)$ such that $\partial(\alpha, \beta)/\partial(\alpha_1, \beta_1) \neq 0$ if $\alpha_1, \beta_1 \in \bar{\Omega}_1$. Moreover*, $m_1 \leq |A| \leq m_2$, $m_1 \leq |B| \leq m_2$, $C = 0$.
 - 4) $\mathbf{r}, \alpha_1, \beta_1$ have continuous derivatives of the third order in Ω ;
 - 5) the external body forces $X, Y, Z \in L_p$, $p > 1$.

Condition 3) is given in order to simplify the presentation of the results and can be greatly weakened.

3. We take as the basis of the theory the relations

$$\varepsilon_1 = \frac{u_{\alpha}}{A} + \frac{A_{\beta}v}{AB} - \frac{w}{R_1} + \frac{w_{\alpha}^2}{2A^2}; \quad \varepsilon_2 = \frac{v_{\beta}}{B} + \frac{B_{\alpha}u}{AB} - \frac{w}{R_2} + \frac{w_{\beta}^2}{2B^2}; \quad (1)$$

$$\omega = \frac{A}{B} \left(\frac{u}{A} \right)_{\beta} + \frac{B}{A} \left(\frac{v}{B} \right)_{\alpha} + \frac{2w}{R_{12}} + \frac{w_{\alpha}w_{\beta}}{AB};$$

$$\chi_1 = \frac{1}{A} \left(\frac{w_{\alpha}}{A} \right)_{\alpha} + \frac{A_{\beta}}{AB^2} w_{\beta}; \quad \chi_2 = \frac{1}{B} \left(\frac{w_{\beta}}{B} \right)_{\beta} + \frac{B_{\alpha}}{A^2B} w_{\alpha}; \quad (2)$$

$$\tau = \frac{w_{\alpha\beta}}{AB} - \frac{B_{\alpha}w_{\alpha}}{AB^2} - \frac{A_{\beta}w_{\beta}}{A^2B}.$$

We take the relation between the forces and the strain components in the form proposed in ⁽⁴⁾ (p. 50). In accordance with this, we simplify the equilibrium equations by Mushtari' s method. As a result, for u, v, w we obtain a system

* The notation is taken from ⁽³⁾.

$$R_1(u, v) = B \left\{ \frac{1}{AB} [(Bu)_{\alpha} + (Av)_{\beta}] \right\}_{\alpha} + \frac{1-\nu}{2} A \left\{ \frac{1}{AB} [(Au)_{\beta} - (Bv)_{\alpha}] \right\}_{\beta} + \frac{1-\nu}{R_1 R_2} u = f_1\{w\}; \quad (3)$$

$$R_2(u, v) = A \left\{ \frac{1}{AB} [(Av)_{\beta} + (Bu)_{\alpha}] \right\}_{\beta} + \frac{1-\nu}{2} B \left\{ \frac{1}{AB} [(Bv)_{\alpha} - (Au)_{\beta}] \right\}_{\alpha} + \frac{1-\nu}{R_1 R_2} v = f_2\{w\}; \quad (4)$$

$$R_3 w = \nabla^4 w = f_3\{u, v, w\}. \quad (5)$$

In (3), (4), (5), $f_1\{w\}$, $f_2\{w\}$ are certain operators in w ; $f_3\{u, v, w\}$ is a certain operator in u, v, w .

For example, let us consider the boundary conditions

$$u|_{\Gamma_{\Omega}} = v|_{\Gamma_{\Omega}} = 0; \quad (6)$$

$$w|_{\Gamma_{\Omega}} = \frac{\partial w}{\partial n} \Big|_{\Gamma_{\Omega}} = 0. \quad (7)$$

4. Let C_1 contain functions w having continuous second derivatives in Ω and satisfying conditions (7). On C_1 define the scalar product

$$(w_1 \cdot w_2)_{H_{1\Omega}} = \int_{\Omega} [\chi_1^{(1)}(\chi_1^{(2)} + \nu\chi_2^{(2)}) + \chi_2^{(1)}(\chi_2^{(2)} + \nu\chi_1^{(2)}) + 2(1-\nu)\tau^{(1)} \cdot \tau^{(2)}] AB d\alpha d\beta. \quad (8)$$

In (8), $\chi_1^{(i)}$, $\chi_2^{(i)}$, $\tau^{(i)}$ are obtained from (2) by substituting $w = w_i$, $i = 1, 2$. The closure of C_1 in the norm (8) will be denoted by $H_{1\Omega}$.

Let C_2 be the set of vector-functions $\vec{\omega}^*(u, v)$ such that u, v have continuous first derivatives in $\bar{\Omega}$ and satisfy conditions (6). On C_2 define the scalar product

$$(\vec{\omega}_1^* \cdot \vec{\omega}_2^*)_{H_{2\Omega}} = \int_{\Omega} \left[\vec{\varepsilon}_1^{(1)}(\vec{\varepsilon}_1^{(2)} + \nu \vec{\varepsilon}_2^{(2)}) + \vec{\varepsilon}_2^{(1)}(\vec{\varepsilon}_2^{(2)} + \nu \vec{\varepsilon}_1^{(2)}) + \frac{1-\nu}{2} \vec{\omega}^{(1)} \cdot \vec{\omega}^{(2)} \right] AB d\alpha d\beta. \quad (9)$$

In (9), $\vec{\varepsilon}_1^{(i)}, \vec{\varepsilon}_2^{(i)}, \vec{\omega}$ are computed by formulas (1), into which $u = u_i, v = v_i, w \equiv 0, i = 1, 2$, are substituted. The closure of C_2 in the norm (9) will be denoted by $H_{2\Omega}$.

By $H_{3\Omega}$ we shall denote the space of vector-functions $\vec{\omega}(u, v, w)$ such that $w \in H_{1\Omega}, \vec{\omega}^*(u, v) \in H_{2\Omega}$. It can be shown that $H_{3\Omega}$ contains vector-functions $\vec{\omega}(u, v, w)$ for which u, v have in Ω first generalized derivatives $^{(5)} \in L_2$; the functions $u, v \in L_q, q \geq 1$, with arbitrary q ; w has in Ω second generalized derivatives $\in L_2$; $w_\alpha, w_\beta \in L_q, q \geq 1$; w is continuous.

Definition. A generalized solution of the system (3)–(5) will mean a vector-function $\vec{\omega}(u, v, w) \in H_{3\Omega}$ satisfying the integral identities

$$(w \cdot \chi)_{H_{1\Omega}} = \int_{\Omega} f_3 \cdot \chi d\alpha d\beta; \quad (10)$$

$$(\vec{\omega}^* \cdot \mathbf{a})_{H_{2\Omega}} + \int_{\Omega} (f_1 \cdot \varphi + f_2 \cdot \psi) d\alpha d\beta = 0, \quad (11)$$

where \mathbf{a} has components φ, ψ , and the vector-function $\mathbf{b}(\varphi, \psi, \chi) \in H_{3\Omega}$ is arbitrary.

Relations (10), (11) write the equilibrium equations of the shell in the form of Lagrange's principle of virtual displacements. This also determines the mechanical meaning of generalized solutions. We note that, in the general case, our system may have several generalized solutions, and this is in full agreement with the mechanical content of the problem. The system (10), (11) can be reduced to a single operator equation in $H_{1\Omega}$. Namely, from (11), $\vec{\omega}(u, v)$ is determined by the functional method in terms of w . After substituting u, v into expression (10), it becomes an operator equation with respect to w . Using the functional method, we replace (10) by the equation $w = Gw$, where Gw is defined by the relation

$$(Gw \cdot \chi)_{H_{1\Omega}} = \int_{\Omega} f_3 \chi d\alpha d\beta, \quad \chi \in H_{1\Omega}$$

and is arbitrary.

Lemma 1. *The operator Gw acts from $H_{1\Omega}$ into $H_{1\Omega}$ and is completely continuous.*

Lemma 1 makes it possible, in order to prove the existence of generalized solutions of the system (3)–(5), to apply the Schauder–Leray principle^(6,7), for which purpose we form in $H_{1\Omega}$ the completely continuous vector field⁽⁷⁾

$$K_1 w = (I - G)w.$$

Lemma 2. *The functional $\Phi(w, t)$, defined by the relation*

$$\Phi(w, t) = \|w\|_{H_{1\Omega}}^2 - t \int_{\Omega} f_3 w \, d\alpha \, d\beta,$$

on spheres of sufficiently large radius R in $H_{1\Omega}$, satisfies the inequality

$$\Phi(w, t) \geq \sigma R^2, \quad (12)$$

where $\sigma > 0$ and does not depend on t , if $0 \leq t \leq 1$.

To prove (12), the sphere S_R of radius R in $H_{1\Omega}$ is divided in a definite way into two parts S_{1R}, S_{2R} , the weak closure of S_{1R} not containing zero. It then turns out that inequality (12) is fulfilled on S_{1R} owing to the growth of the stretching energy of the shell as $\|w\|_{H_{1\Omega}} \rightarrow \infty$, and on S_{2R} owing to the growth of the bending energy.

Lemma 3. *The rotation^(7,8) of the vector field $K_1 w$ on spheres of sufficiently large radius in $H_{1\Omega}$ is equal to +1.*

The proof is based on considering the vector field

$$K_t = (I - tG)w.$$

On spheres of sufficiently large radius in $H_{1\Omega}$, $K_t w$ has no zero vectors if $0 \leq t \leq 1$. Indeed, if for some t_0 the relation $w_0 = t_0 G w_0$ were to hold, then from (10) we would obtain $\Phi(w_0, t_0) = 0$, which is impossible in view of Lemma 2. Thus the fields $K_1 w = (I - G)w$, $K_0 w = Iw$ are homotopic⁽⁷⁾. But the rotation of the field $K_0 w = Iw$ is +1. The lemma is proved. From Lemmas 1, 2 and 3 the following theorems follow.

Theorem 1. *Suppose that all the conditions of item 2 are fulfilled. In this case the system (3)–(7) has at least one generalized solution in the sense of (10)–(11).*

Theorem 2. *If the conditions of item 2 are fulfilled, all generalized solutions of the system (3)–(7) lie in some sphere of the space $H_{1\Omega}$.*

As has already been indicated, the system (10), (11) does not always have a unique solution. In order for uniqueness to hold, additional conditions are necessary.

Theorem 3. *If, under the conditions of item 2, the external forces have the form $X = \lambda X_0$, $Y = \lambda Y_0$, $Z = \lambda Z_0$, then the system (3)–(7) has only one branch of generalized solutions $w(\lambda)$, such that $\|w(\lambda)\|_{H_{1\Omega}} \rightarrow 0$ as $\lambda \rightarrow 0$.*

Theorem 4. Suppose that all the conditions of § 2 are satisfied and, in addition, the inequalities $|1/R_1| \leq \delta$, $|1/R_2| \leq \delta$, $|2/R_{12}| \leq \delta$ hold, where δ is determined by S , the thickness, and the elastic characteristics of the shell. In this case, if X, Y, Z are sufficiently small in the norm in L_p , then (6)–(7) has only one generalized solution.

Let us note that Theorems 3 and 4, at least in broad outline, cover the essential conditions for uniqueness. Indeed, if the shell has sufficiently large curvatures, then even for $X = Y = Z = 0$ several forms of equilibrium are possible. Further, if the curvatures of the shell are small (a plate), then under sufficiently large loads several forms of equilibrium are also possible.

5. Using the equation $\mathfrak{w} = G\mathfrak{w}$, one can study the differential properties of the solutions.
6. The Schauder–Leray method makes it possible to consider other boundary conditions for fixing the shell, including mixed ones, as well as the case of an inhomogeneous and anisotropic shell.

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