THEORY OF ELASTICITY

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Abstract

Full Text

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ON A NONLINEAR THEORY OF THIN PLATES

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In the present work we consider the question of the existence of a solution to the problem of the bending of a thin plate. D. Yu. Panov ([1]), and subsequently Friedrichs and Stoker ([2]), proved the existence of a solution in the case of a circular symmetrically loaded plate. I. I. Vorovich ([3]) considered this question for a shallow shell. We shall solve the problem by methods different from those in ([3]), and for other boundary conditions.

Consider the system of equations

$$\Delta^2 F = \lambda E \left[\left(\frac{\partial^2 w}{\partial x\, \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right],$$

$$\Delta^2 w = \frac{\lambda q}{D} + \frac{\lambda h}{D} \left(\frac{\partial^2 F}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x\, \partial y} \frac{\partial^2 w}{\partial x\, \partial y} \right) \tag{1}$$

with boundary conditions

$$w|_s = 0, \qquad \left.\frac{\partial w}{\partial \nu}\right|_s = 0, \qquad F|_s = 0, \qquad \left.\frac{\partial F}{\partial \nu}\right|_s = 0 \tag{2}$$

or

$$w|_s = 0, \qquad \left.\Delta w - \frac{1-\sigma}{\rho} \frac{\partial w}{\partial \nu}\right|_s = 0, \qquad F|_s = 0, \qquad \left.\frac{\partial F}{\partial \nu}\right|_s = 0. \tag{3}$$

For $\lambda = 1$ the system (1) becomes the well-known system of Kármán equations. For small λ the solution can be found by using various methods of functional analysis, for example the contraction mapping method, Newton' s method as developed by L. V. Kantorovich ([4]), etc. In solving the problem for $\lambda = 1$ we apply the Schauder-Leray method ([5]).

Transforming system (1), we obtain the integral equality

$$\frac{2h}{D}\iint_\Omega F\Delta^2 F\,d\Omega + E\iint_\Omega w\Delta^2 w\,d\Omega = \frac{\lambda E}{D}\iint_\Omega qw\,d\Omega. \tag{4}$$

If F_0 and w_0 belong to W_2^4 ([6]), satisfy (2) or (3) and equation (1), then from (4) it is easy to obtain

$$\|w_0\|_{W_2^2} \le |\lambda| B_1,$$

$$\|F_0\|_{W_2^2} \le |\lambda| B_2. \tag{5}$$

Here, as below, B^i are constants depending only on q, h, D, E, and the contour.

The system of differential equations (1) and boundary conditions (2) or (3) is equivalent to the system of integro-differential equations

$$F = \lambda E \iint_\Omega G(x,y,\xi,\eta)\left[\left(\frac{\partial^2 w}{\partial\xi\partial\eta}\right)^2 - \frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2}\right] d\Omega, \tag{6}$$

$$w = \frac{\lambda}{D}\iint_\Omega Gq\,d\Omega + \frac{\lambda h}{D}\iint_\Omega G\left(\frac{\partial^2 F}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2} + \frac{\partial^2 F}{\partial\eta^2}\frac{\partial^2 w}{\partial\xi^2} - 2\frac{\partial^2 F}{\partial\xi\partial\eta}\frac{\partial^2 w}{\partial\xi\partial\eta}\right) d\Omega,$$

where G is the biharmonic Green' s function satisfying the corresponding boundary conditions.

Differentiating (6) twice with respect to x and y, we obtain a functional equation of the form

$$z = \lambda\Phi(z), \tag{7}$$

where z is a sextuple of functions

$$\frac{\partial^2 w}{\partial x^2},\quad \frac{\partial^2 w}{\partial y^2},\quad \frac{\partial^2 w}{\partial x\partial y},\quad \frac{\partial^2 F}{\partial x^2},\quad \frac{\partial^2 F}{\partial y^2},\quad \frac{\partial^2 F}{\partial x\partial y}.$$

We consider (7) in the Banach vector space of continuous functions. The transformation Φ is completely continuous in the space C. In this case, in order to prove the existence of a solution, we apply the aforementioned Schauder–Leray principle, which consists in the following:

a) The concept of topological degree is introduced ([5]).

b) The transformation $F_\lambda = z - \lambda\Phi(z)$, $\lambda \in [0,1]$, $F_0 = z$, $F_1 = z - \Phi(z)$, is considered.

c) It is shown that F_0 and F_1 are homotopic (^{7}) on some sphere $\|z\|_C = R$ in the space C of continuous vector-functions, and, consequently, the topological degrees of the transformations $F_0(z)$ and $F_1(z)$ at the point zero are equal. Since the degree of $F_0(z)$ at the point zero is known and is equal to $+1$, it follows from homotopy that the degree of $F_1(z)$ at the point zero is also equal to $+1$, and, by the property of the topological degree, there exists in the space C an element $\tilde{z}$ satisfying (5) for $\lambda = 1$.

To prove the homotopy of the transformations F_0 and F_1 on some sphere $\|z\|_C = R$, it is necessary to show that on this sphere there is no point mapped by $F_\lambda(z)$ into zero for any $\lambda \in [0,1]$, or to establish the stronger result that all solutions of the functional equation $z - \lambda\Phi(z) = 0$ for all $\lambda \in [0,1]$ are a priori bounded by some constant B, depending on q, h, D, E and the contour, but not depending on the solutions themselves.

Let us prove the latter assertion. We give some information concerning the biharmonic Green' s function $G(P,Q)$. The function $G(P,Q)$ is continuous and has uniformly continuous first derivatives with respect to both variables in the domain $\overline{\Omega}$. For $P \neq Q$, G is differentiable an arbitrary number of times, and as $P \to Q$ the estimates

$$\left|\frac{\partial^{n+2} G(P,Q)}{\partial P_1^{n_1} \partial P_2^{n_2}}\right| \leq C r_{PQ}^{-n} \ln^2 r_{PQ},$$

$$n_1 + n_2 = n+2, \qquad n = 0, 1, \dots, \qquad P = (P_1 P_2). \tag{8}$$

hold.

Taking into account the estimates (8), it is easy to verify that if z_0 belongs to the space C and satisfies (7), then the corresponding w_0, F_0 belong to W_2^4, satisfy the boundary conditions and equation (1). From (5) it follows:

$$\|z_0\|_{L_2} \leq |\lambda| B_3. \tag{9}$$

But we need to obtain estimates of the form (9) in the space C. The following results are known:

a) If

$$u(P) = \iint_\Omega \ln r_{PQ} \mu(Q)\, d\sigma_Q,$$

where $\mu(Q) \in L_1(\Omega)$ and $\|\mu(Q)\|_{L_1} \leq A_1$, then

$$u(P) \in L_q, \quad q \text{ is arbitrary in } (0,\infty), \quad \|u\|_{L_q} \leq C_1 A_1.$$

b) If, moreover, $\mu(Q) \in L_p$, where $p > 2$, and $\|\mu(Q)\|_{L_p} \leq A_2$, then $u(P) \in C$ and
$$\|u\|_C \leq C_2 A_2.$$

Applying both these results successively to the twice differentiated equations (6), we obtain the inequality

$$\|z_0\|_C \leq |\lambda| B_4. \tag{10}$$

Remark. Since all possible solutions (7) are estimated a priori by inequality (10) through the corresponding λ, applying the contraction-mapping principle, we obtain uniqueness of the solution of (7), and consequently also of (1), for small λ.

Above it was pointed out that estimates are needed for the biharmonic Green' s function and its derivatives. The existence of the Green' s function was shown in the works ([8,9]). The Green' s function is considered as the sum of two terms

$$G(z, z_0) = g(z, z_0) + \frac{r_{zz_0}^2 \ln r_{zz_0}}{2\pi},$$

where

$$\Delta^2 g = 0 \text{ in } \Omega, \qquad g\Big|_S = -\frac{r^2 \ln r}{2\pi}\Bigg|_S, \qquad \frac{\partial g}{\partial \nu}\Bigg|_S = -\frac{\partial \frac{r^2 \ln r}{2\pi}}{\partial \nu}\Bigg|_S;$$

by r, ρ, and r_1 here and below we denote distances between points.

Let $z \in \overline{\Omega}$, $z_0 \in \Omega$. Then the boundary conditions for g are sufficiently smooth, and the apparatus of N. I. Muskhelishvili ([10]) may be applied. We have $\Delta g = 4\,\mathrm{Re}\{\varphi'(z, z_0)\}$. For φ we obtain the integral equation

$$\varphi(\zeta, z_0) - \frac{1}{\pi}\int_S \overline{\varphi}(t, z_0)\, d\theta + \frac{1}{\pi}\int_S \varphi(t, z_0) e^{-2i\theta}\, d\theta = A(\zeta, z_0); \tag{11}$$

$$t - \zeta = re^{i\theta}; \qquad \zeta \text{ is a point of the contour } S;$$

$$A(\zeta, z_0) = \lim_{t_1 \to \zeta,\, t_1 \in \Omega} A(t_1, z_0);$$

$$A(t_1, z_0) = \frac{1}{2\pi i}\int_S \frac{f(t, z_0)}{t - t_1}\, dt; \qquad f(t, z_0) = \frac{\partial \frac{\rho^2 \ln \rho}{2\pi}}{\partial x} + i\frac{\partial \frac{\rho^2 \ln \rho}{2\pi}}{\partial y}\Bigg|_S.$$

Hence

$$\mathrm{Re}\{A(t_1, z_0)\} = \frac{1}{4\pi^2}\int_S \frac{\partial^2 \rho^2 \ln \rho}{\partial s\, \partial y} \ln r_1 \, ds - \frac{1}{4\pi^2}\int_S \frac{\partial \rho^2 \ln \rho}{\partial x}\frac{\partial \ln r_1}{\partial \nu}\, ds, \qquad (12)$$

where $\rho = \rho(t, z_0)$, $r_1 = r_1(t, t_1)$.

Analogously, $\mathrm{Im}\{A(t_1, z_0)\}$ is written.

It is easy to see from (12) that, for a sufficiently smooth contour S and $z_0 \in \Omega$, $\mathrm{Re}\{\varphi(z, z_0)\}$ has n-th continuous derivatives in Ω.

Recall the integral relation existing for the biharmonic Green' s function

$$G(z, z_0) = \frac{r_{zz_0}^2 \ln r_{zz_0}}{2\pi} + \frac{1}{2\pi}\int_S r_{\zeta z}^2 \ln r_{\zeta z} \frac{\partial \Delta G(\zeta, z_0)}{\partial \nu}\, ds - \frac{1}{2\pi}\int_S \frac{\partial r^2 \ln r}{\partial \nu} \Delta G(\zeta, z_0)\, ds. \qquad (13)$$

From the results of N. M. Günter and Ch. L. Smolitskii ([11]) one can derive a lemma.

Lemma. Let

$$u = \int_S \ln r_{x_1 x_2} \ln r_{x_2 x_3}\, ds_2.$$

Then, for a sufficiently smooth contour, for $x_1 \neq x_3$, x_1 and $x_3 \in \Omega$,

$$\left|\frac{\partial^n u}{\partial x_1^n}\right| < B r_{x_1 x_3}^{-n+1} \ln r_{x_1 x_3}, \qquad n = 1, 2, \dots,$$

where $r_{x_1 x_3}$ is the distance between the points x_1 and x_3.

Using the lemma and equality (13), we obtain the theorem:

Theorem. For a sufficiently smooth contour, the biharmonic Green' s function G is uniformly continuous in $\overline{\Omega}$ together with its first derivatives. For $z \neq z_0$ it is differentiable an arbitrary number of times, and as $z \to z_0$ the estimates hold

$$\left|\frac{\partial^{n+2} G(z, z_0)}{\partial z_1^{n_1} \partial z_2^{n_2}}\right| \le C r_{zz_0}^{-n} \ln^2 r_{zz_0}, \qquad n_1 + n_2 = n + 2, \quad z_1 + i z_2 = z, \quad n = 0, 1, \dots$$

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Note: Figure translations are in progress. See original paper for figures.