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Abstract

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ON SOME PROPERTIES OF SERIES IN FABER POLYNOMIALS

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Let $\overline{B}$ be a closed, simply connected, and bounded domain with boundary C; let the function

$$w = \Phi(z) = z + \alpha_0 + \frac{a_1}{z} + \cdots$$

map the exterior of C one-to-one and conformally onto the exterior of the disk $|w| \leq \rho$. The curve in the z-plane which, under the mapping $w = \Phi(z)$, goes into the circle $|w| = r > \rho$ will be denoted by C_r. Consider the system $\{\Phi_n(z)\}$ of Faber polynomials corresponding to the domain B.

Every function $f(z)$, regular in the domain $\overline{B}$, can be expanded in the series

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z), \tag{1}$$

which converges uniformly inside the domain with boundary C_{r_0}, where r_0 is determined by the equality

$$\frac{1}{r_0} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}. \tag{1'}$$

On the boundary C_{r_0} there lies at least one singular point of $f(z)$.

Conversely, the series (1), for which in condition (1′) $r_0 > \rho$, converges uniformly inside the domain bounded by C_{r_0}, and diverges outside it.

Let $f(z)$ be a function regular in the domain $\overline{B}$. Consider all possible orthogonal trajectories to C_r, and along each of them analytically continue the function $f(z)$ from the domain B. The collection of segments of these trajectories along which analytic continuation of the function $f(z)$ is possible, together with the domain B, is called the star of the function $f(z)$ with respect to the domain B. We shall call the generalized star D_r $(r \geq r_0)$ of the function $f(z)$ with respect to the domain B the common part of the star D of the function $f(z)$ with respect to the domain B and the domain bounded by the curve C_r.

In particular, D_∞ will coincide with the star of $f(z)$ with respect to B. If B is a disk, then the star of $f(z)$ is the rectilinear Mittag-Leffler star of the function $f(z)$, and the generalized star is the common part of the rectilinear star and the disk.

Theorem 1. *Changing only the moduli of the coefficients of the series*

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_n(z),$$

we obtain the series

$$\sum_{n=0}^{\infty} a_n \gamma_n \Phi_n(z) \qquad (\gamma_n > 0,\ n = 0, 1, 2, ...),$$

whose sum is a regular function in the generalized star D_r, having C_r as its natural boundary.

Let us note that, in the case when the domain $\overline{B}$ is a disk, and consequently the series (1) is a power series, this theorem was proved by A. I. Seleznev (1).

Proof. Let G be the domain of regularity of the function $f(z)$. It is obvious that $\overline{B}$ is contained inside the domain G. Denote by g the domain onto the exterior of which the complement of the domain G is mapped by means of the function $w = \Phi(z)$. In the domain G take a closed curve L containing the domain $\overline{B}$ inside it, whose image in the w-plane under the function $w = \Phi(z)$ will be l.

Consider the function

$$\varphi(w) = \sum_{n=0}^{\infty} a_n w^n, \tag{2}$$

where a_n are the coefficients of the series (1). The series (2) converges in the disk $|w| < r_0$. As I. F. Lokhin (2) proved, $\varphi(w)$ is a regular function in the domain g, and the function $f(z)$ is represented by the integral

$$f(z) = \frac{1}{2\pi i} \int_L \frac{\varphi|\Phi(\zeta)|}{\zeta - z}\, d\zeta, \tag{3}$$

where z is any point lying inside L; L is a closed rectifiable Jordan arc.

Taking into account, under the indicated conditions, the arbitrariness of the choice of the contour L, we obtain a representation of $f(z)$ by the integral (3) in any closed domain interior to G. We shall prove that the domain g is the domain of regularity of the function $\varphi(w)$. Suppose that g' is the domain of regularity of the function $\varphi(w)$ and $g' \supset g$. Choose in the domain g' a closed curve l', containing the curve l inside it and going beyond the boundary of the domain g. The curve l' in the z-plane corresponds to a curve L', which contains L inside it and goes beyond the boundary of the domain G.

The function $\varphi|\Phi(z)|$ is regular in the domain between the contours L and L' and on them. Then the function

$$F(z) = \frac{1}{2\pi i} \int_{L'} \frac{\varphi|\Phi(\zeta)|}{\zeta - z} \, d\zeta$$

is regular inside L'. On the basis of Cauchy' s theorem we have

$$\int_{L} \frac{\varphi|\Phi(\zeta)|}{\zeta - z} \, d\zeta = \int_{L'} \frac{\varphi|\Phi(\zeta)|}{\zeta - z} \, d\zeta$$

for all z belonging to the domain $\overline{B}$.

Consequently, $f(z) = F(z)$ in the domain $\overline{B}$, whence we obtain that the function $f(z)$ is regular in a domain larger than G, which contradicts the condition.

We have obtained that the domains of regularity of the functions $f(z)$, defined by the series (1), and $\varphi(w)$, defined by the series (2), are "corresponding" under the transformation $w = \Phi(z)$. Then the generalized stars D_r of the function $f(z)$ and d_r of the function $\varphi(w)$ are also "corresponding."

On the basis of the indicated theorem of Seleznev we obtain that for any generalized star d_r of the function $\varphi(w)$ there exists a sequence of positive-

positive numbers $\{\gamma_n\}$ such that the series $\sum_{n=0}^{\infty} a_n \gamma_n w^n$ represents a function whose domain of regularity coincides with d_r. Then, by the correspondence indicated, we obtain that the series $\sum_{n=0}^{\infty} a_n \gamma_n \Phi_n(z)$ will have as its domain of regularity the star D_r.

Consider the series

$$f(z) = \sum_{n=0}^{\infty} a_n \Phi_{\lambda_n}(z) \tag{4}$$

$$\left(\overline{\lim_{n\to\infty}} \, |a_n|^{1/\lambda_n} = \frac{1}{r}, \ r > \rho \right), \qquad 0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots .$$

Theorem 2. If there exists an increasing sequence of natural numbers $\{n_k\}$ such that $\lambda_{n_k+1}/\lambda_{n_k} \to \infty$ as $k \to \infty$, then the series (4) is overconvergent in the domain of regularity of $f(z)$, i.e. the grouped series

$$f(z) = P_{\lambda_{n_1}}(z) + \sum_{k=1}^{\infty} P_{\lambda_{n_k+1}, \lambda_{n_{k+1}}}(z), \tag{5}$$

where

$$P_{\lambda_{n_1}}(z) = a_0 \Phi_{\lambda_0}(z) + \cdots + a_{n_1} \Phi_{\lambda_{n_1}}(z),$$

$$P_{\lambda_{n_k+1}, \lambda_{n_{k+1}}}(z) = a_{n_k+1} \Phi_{\lambda_{n_k+1}}(z) + \cdots + a_{n_{k+1}} \Phi_{\lambda_{n_{k+1}}}(z),$$

converges uniformly inside the domain of regularity of $f(z)$.

Proof. The series

$$\varphi(w) = \sum_{n=0}^{\infty} a_n w^{\lambda_n}, \tag{6}$$

whose coefficients coincide with the coefficients of the series (1), by Ostrowski' s theorem is overconvergent in the domain of regularity of the function $\varphi(w)$. Denote the domains of regularity of the functions $f(z)$ and $\varphi(w)$, respectively, by G and g. Choose an arbitrary closed domain $\overline{D}$ lying inside G, and a contour L lying inside G and enclosing the domain $\overline{D}$. Denote by l the image of the curve L in the w-plane. The grouped power series

$$\varphi(w) = Q_{\lambda_{n_1}}(w) + \sum_{n=1}^{\infty} Q_{\lambda_{n_k+1}, \lambda_{n_{k+1}}}(w)$$

converges uniformly inside and on the contour l. Consequently, the series

$$\varphi[\Phi(\zeta)] = Q_{\lambda_{n_1}}[\Phi(\zeta)] + \sum_{k=1}^{\infty} Q_{\lambda_{n_k+1}, \lambda_{n_{k+1}}}[\Phi(\zeta)] \tag{7}$$

converges uniformly on the contour L.

Multiplying the series (7) by $\frac{1}{2\pi i} \frac{1}{\zeta - z}$ and integrating over the contour L, taking into account formula (3) and the known expression of the Faber polynomials by the integral

$$\Phi_n(z) = \frac{1}{2\pi i} \int_L \frac{[\Phi(\zeta)]^n}{\zeta - z} \, d\zeta, \tag{8}$$

we obtain the uniform convergence of the series (6) in the closed domain.

We note that, taking into account the indicated correspondence between series in Faber polynomials and power series with the same coefficients, one immediately obtains extensions to series in Faber polynomials of, for example, the following theorems: the Hadamard–Ostrowski theorem on gaps, which is obtained from paper (4); Fatou' s theorem on series with a natural boundary, which is obtained from paper (5); Szegő' s theorem on series whose coefficients assume a finite number of values, proved in paper (6).

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Note: Figure translations are in progress. See original paper for figures.