ON SETS OF POINTS AT WHICH THE DERIVATIVE IS EQUAL, RESPECTIVELY, TO \(+\infty\) AND \(-\infty\)

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Abstract

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ON SETS OF POINTS AT WHICH THE DERIVATIVE IS EQUAL, RESPECTIVELY, TO $+\infty$ AND $-\infty$

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In papers (1–4) the descriptive and metric nature was clarified of each of the sets on which the derivative of a function of a real variable is equal, respectively, to $+\infty$ and $-\infty$. In the present note we briefly state a theorem which also resolves the question of the mutual position of these sets.

Theorem. In order that the sets E^1, E^2 of points of the axis OX be the sets of all points at which the derivative of some function of one real variable, finite at every point, exists and is equal, respectively, to $+\infty, -\infty$, it is necessary and sufficient that:

1) E^1, E^2 have measure zero and belong to the class $F_{\sigma\delta}$;
2) there exist two disjoint sets H_1, H_2, belonging to the class F_σ, such that $E^1 \subset H_1,\ E^2 \subset H_2$.

Proof. The necessity of condition 1) is known. Let us construct sets H_1 and H_2 satisfying condition 2).

Let $f(x)$ be a function finite at every point of the axis OX. For any natural n, denote by P_n^1 the set of points x^* for each of which there exists at least one point $z = (x^*, y)$ with abscissa x^*, where $|y| \leq n$, such that

$$\frac{f(x^* + h) - y}{h} \geq 1, \qquad \text{if } 0 < |h| < \frac{1}{n}.$$

Denote by P_m^2 the set of points x^{**} for each of which there exists at least one point $z = (x^{**}, y)$ with abscissa x^{**}, where $|y| \leq m$, such that

$$\frac{f(x^{**} + h) - y}{h} \leq -1, \qquad \text{if } 0 < |h| < \frac{1}{m}.$$

The sets P_n^1 and P_m^2 are closed sets and, for any m and n, do not intersect.

Put

$$H_1 = \sum_n P_n^1, \qquad H_2 = \sum_m P_m^2.$$

We shall now prove the sufficiency of the stated conditions.

1. Let sets E^1, E^2, H_1, H_2 be given, with $\operatorname{mes} E^i = 0,\ H_i \supset E^i,\ H_1 \cdot H_2 = 0$, and

$$E^i = \prod_{n=1}^{\infty} E_n^i, \qquad E_n^i = \sum_{k=1}^{\infty} E_{n,k}^i, \qquad H_i = \sum_{k=1}^{\infty} F_k^i,$$

where $E_{n,k}^i, F_k^i$ are closed sets $(i = 1, 2)$.

Let G_n $(n = 1, 2, ...)$ be open sets containing $E^1 + E^2$, with $\operatorname{mes} G_n < 1/2^n$. We may assume that $G_{n+1} \subset G_n;\ E_n^i \subset G_n;\ F_k^i \subset F_{k+1}^i;\ E_{n,k}^i \subset E_{n,k+1}^i;\ E_{n+1,k}^i \subset E_{n,k}^i \subset F_k^i$. Let C_k^1 and C_k^2 be open disjoint sets, where $C_k^i \supset F_k^i$. Finally, let a summ-

measurable function $u(x) = \sum_{n=1}^{\infty} u_n(x)$, where $u_n(x) = 1$ for $x \in G_n$; $u_n(x) = 0$ for $x \notin G_n$.

Place in the sets E_n^1 and E_n^2, respectively, the sets e_n^1 and e_n^2, where $e_n^i = \sum_{k=1}^{\infty} e_{n,k}^i$, and the $e_{n,k}^i$ are sets simultaneously of type F_σ and G_δ, having the following properties:

1) $e_{n,1}^i = E_{n,1}^i \subset C_1^i$, $e_{n,k+1}^i \supset e_{n,k}^i$, $E^i \cdot E_{n,k}^i \subset e_{n,k}^i \subset E_{n,k}^i \subset C_k^i$.
2) For each integer $k \geqslant 2$ there exist two open sets $g_{n,k}^*$ and $g_{n,k}$, and for $k = 1$ one open set $g_{n,1}$, such that:

a) $G_n = g_{n,1} \supset e_{n,1}^i$;

b) $g_{n,k-1} - \left(e_{n,k-1}^1 + e_{n,k-1}^2\right) \supset g_{n,k}^* \supset g_{n,k} \supset \left(e_{n,k}^i - e_{n,k-1}^i\right)$, where $k \geqslant 2$;

c) the points of the set $e_{n,1}^i$ are points of density for $(g_{n,1} - g_{n,2}^*)$; for $k \geqslant 2$ the points of $(e_{n,k}^i - e_{n,k-1}^i)$ are points of density for $(g_{n,k} - g_{n,k+1}^*)$;

d) if $x_0 \in g_{n,k}^*$, then for any h

$$\frac{\displaystyle\int_{g_{n,k}^h} u(\xi)\, d\xi}{h} < \frac{1}{2^k},$$

where $g_{n,k}^h = g_{n,k} \cdot [x_0, x_0 + h]$ (or $g_{n,k} \cdot [x_0 + h, x_0]$, if $h < 0$).

Obviously, $E_n^i \subset e_n^i \subset \bar{E}_n^i$, and the points of the set e_n^i are points of density for the set $\Omega_n^i = \sum_{k=1}^{\infty} (g_{n,k} - g_{n,k+1}^*) \cdot C_k^i$, moreover $\Omega_n^1 \cdot \Omega_n^2 = 0$.

2. We now construct auxiliary functions.

Let $e^i_{n,1} = \prod_{l=1}^\infty G^{(l)i}_{n,1}$, where $G^{(l)i}_{n,1}$ is an open set, $\overline{G}^{(l+1)i}_{n,1} \subset G^{(l)i}_{n,1} \subset g_{n,1}$. Let $e^i_{n,k} - e^i_{n,k-1} = \prod_{l=1}^\infty G^{(l)i}_{n,k}$, where $G^{(l)i}_{n,k}$ is an open set,

$$G^{(l+1)i}_{n,k} \subset G^{(l)i}_{n,k} \subset g_{n,k}, \quad \overline{G}^{(l+1)i}_{n,k} \subset G^{(l)i}_{n,k} + E^i_{n,k} \quad (n = 1, 2, 3, \dots;\ k = 2, 3, \dots;\ i = 1, 2).$$

We set

$$v^{(1)}_n(x) = \begin{cases} l, & \text{for } x \in G^{(l)1}_n - G^{(l+1)1}_n, \\ +\infty, & \text{for } x \in \prod\limits_{l=1}^\infty G^{(l)1}_n, \\ 0, & \text{at the remaining points;} \end{cases}$$

$$v^{(2)}_n(x) = \begin{cases} -l, & \text{for } x \in G^{(l)2}_n - G^{(l+1)2}_n, \\ -\infty, & \text{for } x \in \prod\limits_{l=1}^\infty G^{(l)2}_n, \\ 0, & \text{at the remaining points,} \end{cases}$$

where $G^{(l)i}_n = \sum_{k=1}^\infty G^{(l)i}_{n,k}$, and, obviously, $e^i_n \subset G^{(l)i}_n \subset G_n$. Let

$$w^{(1)}_n(x) = \begin{cases} \min[v^{(1)}_n(x),\ u(x)] & \text{for } x \in \Omega^1_n, \\ 0 & \text{at the remaining points;} \end{cases}$$

$$w^{(2)}_n(x) = \begin{cases} \max[v^{(2)}_n(x),\ -u(x)] & \text{for } x \in \Omega^2_n, \\ 0 & \text{at the remaining points.} \end{cases}$$

Let, further,

$$W^{(i)}_n(x) = \int_0^x w^{(i)}_n(\xi)\, d\xi.$$

Then:

1) for $x \in E^1_n$ we have $0 \le \overline{W}^{(1)}_n(x) < +\infty$;

2) for $x \in E^2$, moreover, $0 \le \underline{W}^{(1)}_n(x) < +\infty$;

3) for $x \in E^2_n$ we have $-\infty < \underline{W}^{(2)}_n(x) \le 0$;

4) for $x \in E^1$, moreover, $-\infty < \overline{W}^{(2)}_n(x) \le 0$ (by $\overline{W}^{(i)}_n(x)$ and $\underline{W}^{(i)}_n(x)$ are denoted the upper and lower derivatives of the function $W^{(i)}_n(x)$).

3. We pass to the construction of the required function. Put

$$w_n^{(1)*}(x) = \min_{m \le n} w_m^{(1)}(x), \qquad w_n^{(2)*}(x) = \max_{m \le n} w_m^{(2)}(x);$$

$$w_n^{(1)**}(x) = \max[0,\, w_n^{(1)*}(x) - (n-1)];$$

$$w_n^{(2)**}(x) = \min[0,\, w_n^{(2)*}(x) + (n-1)].$$

Let

$$f_n^{(1)}(x) = \min[u_n(x),\, w_n^{(1)*}(x)], \qquad f_n^{(2)}(x) = \max[-u_n(x),\, w_n^{(2)**}(x)].$$

If $x_0 \in E^1$, then x_0 is a point of density for the set where $f_n^{(1)}(x) = 1$, and a point of rarefaction for the set where $f_n^{(2)}(x) = -1$ (these sets do not intersect). If $x^* \in E^2$, then x^* is a point of density for the set where $f_n^{(2)}(x) = -1$, and a point of rarefaction for the set where $f_n^{(1)}(x) = 1$.

Now put

$$f^{(i)}(x) = \sum_{n=1}^{\infty} f_n^{(i)}(x), \qquad F^{(i)}(x) = \int_0^x f^{(i)}(\xi)\, d\xi.$$

The functions $F^{(1)}(x)$ and $F^{(2)}(x)$ satisfy the conditions:

$$\overline{F}^{(1)}(x) \le \overline{W}_n^{(1)}(x) + (n-1), \qquad \underline{F}^{(1)}(x) \le \underline{W}_n^{(1)}(x) + (n-1);$$

$$\overline{F}^{(2)}(x) \ge \overline{W}_n^{(2)}(x) - (n-1); \qquad \underline{F}^{(2)}(x) \ge \underline{W}_n^{(2)}(x) - (n-1).$$

The function absolutely continuous on every segment of the axis OX,

$$F(x) = F^{(1)}(x) + F^{(2)}(x),$$

satisfies the conditions of the theorem:

I. If $x_0 \in E^1$, then $F'(x_0) = +\infty$.

II. If $x^* \in E^2$, then $F'(x^*) = -\infty$.

III. If $x \in \overline{E^1 + E^2}$, then $\underline{F}(x) < +\infty$, $\overline{F}(x) > -\infty$.

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CITED LITERATURE

[1] N. N. Luzin, *Collected Works*, 1, 1953, p. 5. [2] Z. S. Zagorskii, *Matem. sborn.*, **9** (51), 3, 487 (1941). [3] A. L. Brudno, *Matem. sborn.*, **13** (55); 1, 119 (1943). [4] E. M. Landis, *DAN*, **107**, No. 2 (1956).

Note: Figure translations are in progress. See original paper for figures.

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