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**Abstract**

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Mathematics

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# ON LINEAR INTERPOLATION OF STATIONARY PROCESSES WITH DISCRETE TIME

*(Presented by Academician A. N. Kolmogorov on 16 IV 1957)*

In this note, the condition of interpolability of a stationary sequence due to A. N. Kolmogorov,

$$\int_{-\pi}^{\pi} \frac{d\lambda}{f_\lambda} = \infty \qquad (*)$$

and the results of work [(2)] are generalized to multidimensional and infinite-dimensional processes. In § 1 a coordinate-free method is set forth for defining the very concept of a stationary process, which in § 2 leads to a more elegant formulation of the results.

§ 1. **A stationary process and its spectral functional.** In this paragraph we present jointly the cases of continuous time ($t$ ranges over all real values) and discrete time ($t$ ranges over all integer values).

In the coordinate approach, an $n$-dimensional stationary process is a collection of $n$ random complex functions

$$x_1(t), \quad x_2(t), \dots, x_n(t) \qquad (1)$$

such that

$$Mx_i(t) = m_i, \qquad Mx_i(t+\tau)\overline{x_j(t)} = B_{ij}(\tau); \qquad (2)$$

here, without loss of generality, one takes $m_i = 0$.

Consider $H$, the collection of random complex variables $x$ with $Mx = 0$ and $M|x|^2 < \infty$. If all $x \in H$ that differ from one another only with probability equal to 0 are identified, then $H$ becomes a Hilbert space with scalar product $(x, y) = Mx\bar{y}$. To each vector $a = (a^1, a^2, \dots, a^n)$ of the affine $n$-dimensional space $A$ we associate the element of the space $H$

$$x_t(a) = a^1 x_1(t) + a^2 x_2(t) + \dots + a_n^{nx}(t).$$

It is easy to see that the scalar product

$$B_\tau(a,b) = (x_{t+\tau}(a), x_t(b)) \tag{3}$$

does not depend on $t$ for any $a, b \in A$.

Obviously, the study of the system of functions (1) is equivalent to the study of the function $x_t(a)$ of $t$ and $a$.

In the space $A$ it is natural to introduce the scalar product $(a,b) = B_0(a,b)$ and the norm $\|a\| = (a,a)^{1/2}$, and to identify the elements $a$ and $b$ in the case $\|a-b\| = 0$. Then $A$ becomes a unitary space, and the operators $x_t$ turn out to be isometric.

The coordinate-free conception of stationary processes, proposed by A. N. Kolmogorov and used below, consists in the following. A **stationary process** $\{A, x_t\}$ is a collection consisting of a unitary space $A$ of "simultaneously observable quantities" and linear isometric operators $x_t$ from $A$ into $H$, satisfying the condition that the scalar products (3) are independent of $t$. It is natural to impose on $x_t$ the continuity condition

$$\lim_{t-s\to 0} \|x_t(a) - x_s(a)\| = 0$$

for all $a \in A$.

Let $H_x$ be the linear closure of the elements $x_t(a)$ in $H$. The equalities $U^\tau x_t(a) = x_{t+\tau}(a)$ define on $H_x$ [(1)] unitary operators $U^\tau$,

$$U^{tU^\tau} = U^{t+\tau}, \quad \|U^{t+\tau}x - U^\tau x\| \to 0 \quad \text{as } t \to 0$$

and for every $x \in H$.

For $U^\tau$ the representation

$$U^\tau = \int e^{i\lambda\tau}\, dE_\lambda, \tag{4}$$

holds, where $E_\lambda$ is a spectral family (here and below the integration is over $-\pi \le \lambda \le \pi$ in the discrete case, and over $-\infty < \lambda < +\infty$ in the continuous case).

We shall call the bilinear functional

$$F_\lambda(a,b) = (E_\lambda x_0(a),\, x_0(b)). \tag{5}$$

the **spectral functional** of the process. Obviously,

$$B_\tau(a,b) = \int e^{i\lambda\tau}\, dF_\lambda(a,b).$$

For fixed $a$ and $b$, the spectral functional $F_\lambda(a,b)$ almost everywhere has a derivative

$$f_\lambda(a,b) = F'_\lambda(a,b), \tag{6}$$

which we shall call the **spectral density**. Let $M_\lambda$ be the set of $a \in A$ for which $f_\lambda(a,a)$ exists; every $a \in A$ belongs to $M_\lambda$ for almost all $\lambda$; $f_\lambda(a,b)$ is a bilinear functional on $M_\lambda$, which, in the case of separability of $A$, is everywhere dense in $A$ for almost all $\lambda$.

§ 2. **On linear interpolation.** We shall consider the discrete-time case. Let $T$ be a finite set of integers, and let $H(T)$ and $\widehat{H}(T)$ be, respectively, the linear closures of $x_t(a)$ for $t \in T,\ a \in A$, and of $x_t(a)$ for $t \bar{\in} T,\ a \in A$. The question is: when can $x_t(a)$, $t \in T$, be linearly interpolated, knowing $x_t(a)$ for $t \bar{\in} T$, i.e. when is $H(T) \subseteq \widehat{H}(T)$?

With respect to the process $\{A, x_t\}$ we shall assume that $A$ is separable and $f_\lambda$ is bounded for almost all $\lambda$, i.e. $|f_\lambda(a,b)| \le C_\lambda \|a\|\,\|b\|$, $a, b \in M_\lambda$.

Then $f_\lambda(a,b) = (f_\lambda a, b)$, where $f_\lambda$ is some positive bounded linear operator on $A$.

Let $\theta(\lambda)$ be the subspace of elements $a \in A$ for which $f_\lambda a = 0$; $A(\lambda) = A \ominus \theta(\lambda)$; on $A(\lambda)$ there exists $f_\lambda^{-1}$.

Consider the Hilbert space $\mathfrak{B}(T)$ of functions $b_\lambda$, defined almost everywhere on $[-\pi, \pi]$ and such that:

$$1^\circ.\quad b_\lambda = \sum_{t \in T} e^{i\lambda t} b_t, \quad b_t \in A.$$

$$2^\circ.\quad b_\lambda \in f_\lambda A.$$

$$3^\circ.\quad \int_{-\pi}^{\pi} (f_\lambda^{-1} b_\lambda, b_\lambda)\, d\lambda < \infty.$$

The scalar product in $\mathfrak{B}(T)$ is

$$\int_{-\pi}^{\pi} (f_\lambda^{-1} b_\lambda,\ b'_\lambda)\, d\lambda.$$

**Main theorem.** $\Delta(T) = H(T) \ominus \widehat{H}(T)$ is isometric to $\mathfrak{B}(T)$.

Let, for example, $A$ be an $n$-dimensional unitary space; $a_1, a_2, \dots, a_n$ an orthonormal basis in it; $x_i(t) = x_t(a_i)$. Then $f_\lambda$ is given by the matrix $\|f_{ij}(\lambda)\|$ of ordinary spectral densities:

$$f_{ij}(\lambda) = \frac{d}{d\lambda}(E_\lambda x_i(0),\, x_j(0)).$$

The function $b_\lambda$ is the vector $(b^1_\lambda, \dots, b^n_\lambda)$, where $b^k_\lambda = \sum_{t\in T} e^{i\lambda t}\alpha_t$ is a trigonometric polynomial; conditions 2° and 3° for $b_\lambda$ look quite simple; for example, if $f_\lambda = \|f_{ij}(\lambda)\|$ is non-singular for almost all $\lambda$, then 2° is satisfied, and 3° is

$$\int_{-\pi}^{\pi} \sum_{i,j=1}^{n} p_{ij}(\lambda) b^i_\lambda \overline{b^j_\lambda}\, d\lambda < \infty, \quad \text{where } \|p_{ij}(\lambda)\| = f_\lambda^{-1}.$$

For $n = 1$ one easily obtains the result of A. M. Yaglom (2): $\Delta(T) \neq 0$ if and only if there exists a set of numbers $\alpha_t$, $\sum_{t\in T} |\alpha_t|^2 \neq 0$, and

$$\int_{-\pi}^{\pi} \frac{\left|\sum_{t\in T} e^{i\lambda t}\alpha_t\right|^2}{f_\lambda}\, d\lambda < \infty. \tag{7}$$

Let us further note that if the number of elements of $T$ is $s$, then the dimension $\Delta(T) \le ns$; in order that the equality sign hold, it is necessary and sufficient that

$$\int_{-\pi}^{\pi} \operatorname{sp} f_\lambda^{-1}\, d\lambda < \infty, \qquad \operatorname{sp} f_\lambda^{-1} = \sum_{i=1}^{n} p_{ii}(\lambda). \tag{8}$$

For $n = 1,\ s = 1$ one obtains the result of A. N. Kolmogorov (*), contained in (1).

For the proof of the main theorem we shall need the following lemma.

**Lemma.** Let $F_\lambda(a, b)$, for arbitrary $a, b \in A$, be an absolutely continuous function. Then $H_x$ is isometric to the space $\mathfrak{A}$ of functions $a_\lambda \in A(\lambda)$, defined almost everywhere on $[-\pi, \pi]$, $(a_\lambda, a)$ measurable* for every $a \in A$ and

$$\int_{-\pi}^{\pi} (a_\lambda, a_\lambda)\, d\lambda < \infty;$$

the scalar product in $\mathfrak{A}$ is

$$\int_{-\pi}^{\pi} (a_\lambda, a'_\lambda)\, d\lambda;$$

moreover all $a_\lambda$ equal to one another almost everywhere are identified.

The assertion of the lemma follows from the fact that the $\mathfrak{A}$-complete space, and the linear combinations of the functions $e^{i\lambda t} f_\lambda^{1/2} a$ ($f_\lambda^{1/2}$ is the positive square root of $f_\lambda$, $a \in A$, $-\infty < t < +\infty$) are everywhere dense in $\mathfrak{A}$. Namely,

$$x_t(a) \leftrightarrow e^{i\lambda t} f_\lambda^{1/2} a \tag{9}$$

generates an isometric correspondence between $H_x$ and $\mathfrak{A}$.

**Proof of the main theorem.** It is not difficult to show that, without loss of generality, one may assume the spectral functional $F_\lambda(a, b)$ to be absolutely continuous for arbitrary $a$ and $b$. Then, by the lemma, $H_x$ is isometric to $\mathfrak{A}$. $\Delta(T)$ is a subspace of $H_x$, and it corresponds to some subspace $\mathfrak{A}(T) \subset \mathfrak{A}$.

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* Concerning the definition of measurability of $a_\lambda$, see, for example, ([3]), p. 54.

Take $\delta \in \Delta(T)$ and the corresponding $a_\lambda \in \mathfrak{A}$; $(\delta, x_t(a)) = 0$ for $t \notin T$, $a \in A$. We have (see (9))

$$(\delta, x_t(a)) = \int_{-\pi}^{\pi} (a_\lambda, e^{i\lambda t} f_\lambda^{1/2} a)\, d\lambda = \int_{-\pi}^{\pi} e^{-i\lambda t} (f_\lambda^{1/2} a_\lambda, a)\, d\lambda = 0,$$

$$(f_\lambda^{1/2} a_\lambda, a) = \sum_{t \in T} e^{i\lambda t} \alpha_t(a),$$

where

$$\alpha_t(a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda t} (f_\lambda^{1/2} a_\lambda, a)\, d\lambda$$

is a bounded linear functional, since

$$|\alpha_t(a)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} (a_\lambda, a_\lambda)^{1/2} (f_\lambda a, a)^{1/2}\, d\lambda \le$$

$$\le \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} (a_\lambda, a_\lambda)\, d\lambda \right)^{1/2} \left( \int_{-\pi}^{\pi} (f_\lambda a, a)\, d\lambda \right)^{1/2} = \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} (a_\lambda, a_\lambda)\, d\lambda \right)^{1/2} \|a\|.$$

Hence

$$\alpha_t(a) = (b_t, a), \qquad b_t \in A,$$

$$(f_\lambda^{1/2} a_\lambda - \sum_{t\in T} e^{i\lambda t} b_t, a) = 0, \qquad f_\lambda^{1/2} a_\lambda = \sum_{t\in T} e^{i\lambda t} b_t.$$

Put $f_\lambda^{1/2} a_\lambda = b_\lambda$. We have

$$\int_{-\pi}^{\pi} (a_\lambda, a_\lambda)\, d\lambda = \int_{-\pi}^{\pi} (f_\lambda^{-1} b_\lambda, b_\lambda)\, d\lambda < \infty.$$

Conversely, if

$$b_\lambda \in f_\lambda A, \qquad \int_{-\pi}^{\pi} (f_\lambda^{-1} b_\lambda, b_\lambda)\, d\lambda < \infty, \qquad b_\lambda = \sum_{t\in T} e^{-i\lambda T} b_t,$$

then

$$a_\lambda = f_\lambda^{-1/2} b_\lambda \in \mathfrak{A}(T)$$

and determines some element $\delta \in \Delta(T)$.

Thus, $\Delta(T)$ is isometric to $\mathfrak{B}(T)$, as was required to prove.

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## REFERENCES

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2. A. M. Yaglom, *Uspekhi Mat. Nauk*, **4**, No. 4, 171 (1949).

3. E. Hille, *Functional Analysis and Semigroups*, IL, 1951.

*Note: Figure translations are in progress. See original paper for figures.*