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1957

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Abstract

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Reports of the Academy of Sciences of the USSR

1957. Volume 115, No. 6

PHYSICS

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ON THE FUNCTION OF THE SPATIAL DISTRIBUTION OF PARTICLES IN AN ELECTRON-PHOTON SHOWER

(Presented by Academician D. V. Skobeltsyn on 9 IV 1957)

In paper (1) a method was proposed for calculating the functions of the angular and spatial distribution of electrons at the maximum of a cascade shower. In the present paper a method is proposed for computing a broader class of functions from their known moments.

Let us consider a function $\Phi(r, s)$, depending on the parameter s (and a number of other parameters), defined on the interval $0 \leq r < \infty$, and suppose it is known that as $r \rightarrow 0$

$$\Phi(r, s) = \frac{1}{r^{2-s}} \varphi(r, s); \tag{1}$$

here $\varphi(r, s) = \text{const} \neq 0$ as $r \rightarrow 0$; $0 < s \leq 2$.

The moments with respect to r of the function $\Phi(r, s)$, defined by the formula

$$\overline{r^n}_{\Phi}(s) = \int_0^{\infty} \Phi(r, s) r^n dr / \int_0^{\infty} \Phi(r, s) r dr, \tag{2}$$

will be regarded as known. It is easy to see that

$$\overline{r^n}_{\Phi}(s) = \overline{r^{n+s-1}}_{\varphi}(s), \tag{3}$$

where

$$\overline{r^m}_{\varphi}(r) = \int_0^{\infty} \varphi(r, s) r^m dr / \int_0^{\infty} \varphi(r, s) dr$$

is the m -th moment of the function $\varphi(r, s)$.

We note that in all practically interesting cases it is not possible to obtain an explicit analytic expression for the moments $\overline{r^n_\Phi}(s)$. Usually it is possible to obtain numerical values only for the first few moments of the sought distribution function. Therefore, for computing the function $\Phi(r, s)$ from its moments, it is practically more expedient to approximate it by some other function, the first several moments of which coincide with the exact moments of the desired function.

We approximate the function $\varphi(r, s)$ by means of a sum of polynomials $R_n(\alpha r, s)$:

$$\varphi(r, s) = e^{-\alpha r^\beta} \sum_{n=0}^N a_n(s) R_n(\alpha r, s). \quad (4)$$

The polynomials $R_n(\alpha r, s)$ are orthogonal on the interval $[0, \infty]$ with the weight function $e^{-\alpha r^\beta}$. We determine the coefficients $a_n(s)$ from the condition of orthogonality of the polynomials, by the formula

$$a_n(s) = \int_0^\infty \varphi(r, s) R_n^+(r, s) dr / \int_0^\infty e^{-\alpha r^\beta} R_n(\alpha r, s) R_n^+(r, s) dr. \quad (5)$$

The coefficients $a_n(s)$ are expressed in terms of the moments of the sought distribution function. It is natural to assume that the n -th coefficient is a linear combination of the first n nonzero moments; therefore the polynomial $R_n^+(r, s)$ (taking relation (3) into account) is expressed by the formula

$$R_n^+(r, s) = \sum_{m=0}^n c_m(s) r^{m+s-1}.$$

From this it is easy to write the conditions determining the polynomials $R_n(\alpha r, s)$ and $R_n^+(r, s)$:

$$\int_0^\infty r^{s+kn'-1} e^{-\alpha r^\beta} R_n(\alpha r, s) dr = \begin{cases} 0, & \text{for } n' < n, \\ \text{const} \neq 0, & \text{for } n' = n; \end{cases} \quad (6a)$$

$$\int_0^\infty e^{-\alpha r^\beta} R_{n'}(\alpha r, s) R_n^+(r, s) dr = \delta_{n, n'}. \quad (6b)$$

Here $\delta_{n, n'}$ is the Kronecker symbol; k is a certain constant equal to an integer. The coefficients α and β are determined from the asymptotic behavior of $\varphi(r, s)$ as $r \rightarrow \infty$. If the behavior of $\varphi(r, s)$ as $r \rightarrow \infty$ is unknown, then α and β are determined from the condition that the two highest moments used in (4) of the

weight function be equal to the corresponding moments of the sought function, i.e.

$$\int_0^\infty e^{-\alpha r^\beta} r^k dr / \int_0^\infty e^{-\alpha r^\beta} dr = \overline{r^k}. \quad (7)$$

Let us note that for $\beta = 1$, $k = 1$, $s = 2$, conditions (6) determine polynomials coinciding with the Laguerre polynomials. Taking α equal to the minimum value of the total photon absorption coefficient $\sigma_{\min}(E)$, we obtain that expansion (4) coincides with the approximation formula used in works (2) in calculating the dependence of the number of electrons and photons on depth and energy in a cascade shower in light and heavy substances. It appears to us that the proposed method of approximating functions from their known moments (see equations (4), (5), (6), and (7)) can be successfully applied in solving a wide range of problems. In the present work we shall apply this method to the calculation of the functions of the spatial distribution of particles in an electron-photon shower. It is known (see, for example, (3)) that the distribution function $N_p(E_0, E, t, r) dt dr$, giving the number of electrons with energy greater than E at depth t , $t + dt$, traveling at a distance r , $r + dr$ from the axis*, in a shower produced by a primary particle with energy E_0 , as $r \rightarrow 0$ is proportional to

$$N_p(E_0, E, t, r) \sim \frac{1}{r^{2-s}}, \quad (8)$$

where s is the cascade parameter.

We introduce the function

$$N_p(E, r, s) = N_p(E_0, E, t, r) / N_p(E_0, E, t),$$

where $N_p(E_0, E, t)$ is the function describing the one-dimensional development of the shower (4). It is not difficult to show that $N_p(E, r, s) = N_p(x, s)$, where $x = Er/E_s$, $E_s = 21$ MeV. Taking (8) into account, we represent $N_p(x, s)$ in the form

$$N_p(x, s) = \frac{1}{x^{2-s}} f(x, s). \quad (9)$$

Using formulas (5), (4), (6), we shall calculate the function $f(x, s)$, assuming the moments of the function $N_p(E_0, E, t, r)$ to be known. As a result we obtain the following expressions for the polynomials $R_n(\alpha x, s)$ and $R_n^+(x, s)$:

* By the axis of the shower we mean the straight line continuing the direction of motion of the primary particle that produced the shower.

$$R_0(\alpha x, s) = \frac{\beta}{\varphi_0(s)} \alpha^{s/\beta}; \quad R_1(\alpha x, s) = \frac{\beta}{\varphi_1(s)} \alpha^{(s+2)/\beta} [-\Gamma_1 + \varphi_0(s) \alpha^{1/\beta-1} \alpha x]; \quad (10)$$

$$R_2(\alpha x, s) = \frac{\beta}{\varphi_2(s)} \alpha^{(s+4)/\beta} [(\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3) - (\Gamma_0 \Gamma_4 - \Gamma_2^2) \alpha^{1/\beta-1} \alpha x + \varphi_1(s) \alpha^{2/\beta-2} (\alpha x)^2],$$

where

$$\Gamma_n = \Gamma \left(\frac{s+n}{\beta} \right); \quad \varphi_0(s) = \Gamma_0; \quad \varphi_1(s) = \Gamma_0 \Gamma_3 - \Gamma_1 \Gamma_2; \quad \varphi_2(s) = \Gamma_0 (\Gamma_3 \Gamma_6 - \Gamma_4 \Gamma_5) - \\ - \Gamma_2 (\Gamma_1 \Gamma_6 - \Gamma_2 \Gamma_5) + \Gamma_4 (\Gamma_1 \Gamma_4 - \Gamma_2 \Gamma_3);$$

$$R_0^+(x, s) = x^{s-1}; \quad R_1^+(x, s) = -\frac{\Gamma_2}{\varphi_0(s)} \alpha^{-2/\beta} x^{s-1} + x^{s+1};$$

$$R_2^+(x, s) = \frac{1}{\varphi_1(s)} (\Gamma_5 \Gamma_2 - \Gamma_3 \Gamma_4) \alpha^{-4/\beta} x^{s-1} + \frac{1}{\varphi_1(s)} (\Gamma_1 \Gamma_4 - \Gamma_0 \Gamma_5) \alpha^{-2/\beta} x^{s+1} + x^{s+3}. \quad (11)$$

Using formulas (5), (10), (11), we obtain expressions for $a_n(s)$:

$$a_0(s) = 1; \quad a_1(s) = -\frac{\Gamma_2 \alpha^{-2/\beta}}{\varphi_0(s)} + \overline{r^2}; \\ a_2(s) = \frac{\Gamma_5 \Gamma_2 - \Gamma_3 \Gamma_4}{\varphi_1(s)} \alpha^{-4/\beta} - \frac{\Gamma_1 \Gamma_4 - \Gamma_0 \Gamma_5}{\varphi_1(s)} \alpha^{-2/\beta} \overline{r^2} + \overline{r^4}, \dots \quad (12)$$

The coefficients α and β were determined from condition (7), and for β values ~ 0.5 were obtained. In paper (5) it is shown that for $r \rightarrow \infty$ the function $N_p(E_0, E, r, t) \sim e^{-\alpha \sqrt{x}}$; therefore, in the numerical calculations we took $\beta = 0.5$, while α was determined from (7). The quantity N in (4) was taken equal to $n/2$, where n is the number of the highest moment used in the expansion.

By a method analogous to that used in papers (6), we obtained recurrent algebraic relations for computing the n -th angular and spatial moments of the particle distribution functions without allowing for ionization losses. From these formulas numerical values of $\overline{r^2}$, $\overline{r^4}$, $\overline{r^6}$ were obtained for the spatial distribution

function of electrons in the interval of values of the cascade parameter s from 0.4

Table 1

x	$s = 0,4$	$s = 0,6$	$s = 0,8$	$s = 1,1$	$s = 1,3$	$s = 1,5$	$s = 1,7$	$s = 1,9$
0	1,72	4,04	7,34	$1,21 \cdot 10^1$	$1,38 \cdot 10^1$	$1,36 \cdot 10^1$	$1,21 \cdot 10^1$	9,87
0,0001	1,59	3,79	6,96	1,16	1,33	1,32	1,17	9,58
0,0002	1,55	3,70	6,82	1,14	1,31	1,30	1,16	9,46
0,0005	1,46	3,52	6,52	1,10	1,27	1,26	1,13	9,23
0,001	1,37	3,33	6,21	1,06	1,22	1,22	1,09	8,98
0,002	1,25	3,08	5,80	1,00	1,16	1,16	1,05	8,62
0,005	1,06	2,64	5,08	8,93	1,05	1,06	9,62	7,97
0,01	$8,94 \cdot 10^{-1}$	2,24	4,36	7,87	9,35	9,56	8,73	7,29
0,015	7,91	1,98	3,89	7,13	8,56	8,82	8,12	6,81
0,02	7,17	1,80	3,53	6,56	7,93	8,25	7,62	6,43
0,04	5,42	1,32	2,63	5,08	6,28	6,68	6,28	5,36
0,06	4,43	1,08	2,10	4,16	5,24	5,67	5,41	4,67
0,1	3,24	$7,48 \cdot 10^{-1}$	1,48	3,04	3,93	4,37	4,23	3,74
0,15	2,36	5,39	1,04	2,21	2,94	3,36	3,34	2,99
0,2	1,79	4,09	$7,87 \cdot 10^{-1}$	1,69	2,31	2,69	2,72	2,47
0,6	$3,20 \cdot 10^{-2}$	$9,26 \cdot 10^{-2}$	1,55	$3,81 \cdot 10^{-1}$	$6,05 \cdot 10^{-1}$	$7,65 \cdot 10^{-1}$	$8,65 \cdot 10^{-1}$	$8,57 \cdot 10^{-1}$
1,0	$6,55 \cdot 10^{-3}$	2,79	$5,75 \cdot 10^{-2}$	1,33	2,16	3,10	3,80	4,03
1,5	1,88	$8,18 \cdot 10^{-3}$	1,94	$4,59 \cdot 10^{-2}$	$7,71 \cdot 10^{-2}$	1,22	1,56	1,84
2,0	$5,08 \cdot 10^{-4}$	2,71	$7,52 \cdot 10^{-3}$	1,85	3,21	$5,39 \cdot 10^{-2}$	$7,76 \cdot 10^{-2}$	$9,44 \cdot 10^{-2}$
2,5	1,54	$9,73 \cdot 10^{-4}$	3,17	$8,29 \cdot 10^{-3}$	1,41	2,55	3,95	5,12
3,0	$5,12 \cdot 10^{-5}$	3,67	1,42	4,00	$6,80 \cdot 10^{-3}$	1,27	2,10	2,90
3,5	1,85	1,46	$6,61 \cdot 10^{-4}$	2,04	3,32	$6,50 \cdot 10^{-3}$	1,15	1,73
4,0	$7,19 \cdot 10^{-6}$	$5,81 \cdot 10^{-5}$	3,16	1,11	1,75	3,40	$6,44 \cdot 10^{-3}$	1,01
4,5	3,03	2,37	1,54	$6,30 \cdot 10^{-4}$	$9,23 \cdot 10^{-4}$	1,81	3,65	$6,14 \cdot 10^{-3}$
5,0	1,38	1,01	$7,59 \cdot 10^{-5}$	3,72	5,26	$9,79 \cdot 10^{-4}$	2,08	3,77

x	$s = 0,4$	$s = 0,6$	$s = 0,8$	$s = 1,1$	$s = 1,3$	$s = 1,5$	$s = 1,7$	$s = 1,9$
5,5	$6,81 \cdot 10^{-7}$	$4,42 \cdot 10^{-6}$	3,76	2,28	3,04	5,25	1,19	2,33
6,0	3,59	2,08	1,86	1,45	1,92	2,87	$6,74 \cdot 10^{-4}$	1,45

to 2.0*, used by us in the calculations by formula (4).

The obtained values of the spatial-distribution function $x^{2-s}N_p(x, s)**$ are given in Table 1. The values $N_p(x, s)$ were obtained using $\bar{r}^2, \bar{r}^4, \bar{r}^6$.

The series in moments (4) determines a function for which only the first few moments coincide with the exact ones. The higher moments of this function will differ from the exact ones. However, taking into account that the asymptotic behavior of (4) coincides with the asymptotic behavior of the exact function as $x \rightarrow \infty$, one may expect that the higher moments of the function (4) will not differ greatly from the exact ones, and that the function (4) differs from the exact one by no more than a few percent over a broad range of values of x . Using the expression $N_p(x, s)$, calculated from 3 even moments, we computed \bar{r}^8 and \bar{r}^{10} for $s = 1$. It turned out that the computed values of \bar{r}^8 and \bar{r}^{10} agree with the exact ones within 10%. It is natural to suppose that the values of \bar{r}^n are determined mainly by some finite interval of values of x , depending on n . Therefore, by comparing the values $N_p(x, s)$ constructed, for example, from the first 2 and 3 even moments of the function, one can determine the limits of applicability in x of approximation (4). Studies show that even when only the first 2 moments are used, formula (4) makes it possible to calculate $N_p(x, s)$ in the interval of s from ~ 0.4 to ~ 2 and x from 0 to ~ 6 with an error not exceeding 10%.

In work (6), from the moments, the functions $N_p(x, s)$ were constructed for $s = 0.6; 1; 1.5$ by the method of graphical fitting of an arbitrary function whose first few moments coincide with the exact ones. In constructing $N_p(x, s)$, the authors made substantial use of Molière's calculation results (8) for the spatial-distribution function, exact at small x for $s = 1$. The comparison carried out in (1) of the functions $N_p(x, s)$, obtained in (6) and calculated by our method for $s = 1$, showed that the functions agree within 10%. In constructing $N_p(x, s)$ for $s = 0.6$ and 1.5, the authors of (6) assumed that the character of the behavior of $N_p(x, s)$ as $x \rightarrow 0$ differs little from the behavior of $N_p(x, s)$ as $x \rightarrow 0$ for $s = 1$, and they considered that the nature of the singularity at $x \rightarrow 0$ for the given s does not affect the behavior of the function for $x > 0.2-0.4$.

Comparison of the functions $N_p(x, s)$ for $s = 0.6$ and 1.5, taken from (6) and calculated by (4), shows that they differ substantially (by about a factor of 2) from one another. At the same time, the functions constructed by (4) for $s = 0.6$

and 1.5 under the assumption that $N_p(x, s) \underset{x \rightarrow 0}{\sim} 1/x$ agree much better with the functions obtained in (6).

The authors express their gratitude to N. S. Strunnina for assistance in carrying out the numerical calculations.

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Received
2 IV 1957

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* We note that the values $\overline{\theta^2}$, $\overline{\theta r}$, and $\overline{r^2}$ calculated by us for s from 1.1 to 1.9 differ by several percent from the values obtained by Borsellino (7).

** The function $N_p(x, s)$ is normalized as follows:

$$\int_0^{\infty} N_p(x, s) x dx = 1.$$

Note: Figure translations are in progress. See original paper for figures.

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