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Abstract

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ON THE COMPOSITION OF THE PROBABILITY LAWS OF GAUSS AND POISSON

In the present note we briefly set out a proof of the following theorem, which generalizes the well-known theorems of H. Cramér (1) and D. A. Raikov (2).

Theorem. *A composition of the laws of Gauss and Poisson can be decomposed only into similar compositions; moreover, the sum of the variances of the Gaussian summands is equal to the variance of the principal Gaussian component, and the same holds for the Poisson summands.*

Let us make the formulation of the theorem more precise. A random variable X will, as usual, be called Poisson if $P(X < x)$ has the form $F\left(\frac{x-\alpha}{\beta}; \lambda\right)$, where α is any real number; $\beta > 0$; $\lambda \geqslant 0$;

$$F(x; \lambda) = e^{-\lambda} \sum_{k \leqslant x} \frac{\lambda^k}{k!} \quad (x \geqslant 0).$$

Let X_1 be a normal random variable; X_2 a Poisson random variable; X_2 is independent of X_1, and

$$Y = X_1 + X_2.$$

Suppose that we have some other decomposition

$$Y = Y_1 + Y_2,$$

where Y_1 and Y_2 are independent random variables. Then

$$Y_j = Y_{j1} + Y_{j2} \quad (j = 1, 2),$$

where Y_{j1} are normal and Y_{j2} are Poisson $(j = 1, 2)$, and

$$D(Y_{1j}) + D(Y_{2j}) = D(X_j).$$

The proof of this theorem is carried out under the assumption that $D(X_1) > 0$, $D(X_2) > 0$, and does not rely on the theorems of H. Cramér and D. A. Raikov. The latter are derived from it almost immediately if one takes $D(X_1) \to 0$ or $D(X_2) \to 0$.

The proof of the theorem is at present rather complicated and is based on many facts from the theory of entire functions and of measurable and Lebesgue-summable functions of a real variable. One analytic lemma of I. M. Vinogradov (“I. M. Vinogradov’ s cups” *) plays a useful role in it. Let us outline the main stages of the proof. Without loss of generality we may assume that:

$$E \exp(zY) = \exp(\gamma z^2 + \lambda(e^z - 1)) = \varphi(z), \tag{1}$$

where $z = x + iy$, $\gamma > 0$, $\lambda > 0$. It is known (²) that $E \exp(zY_1) = \varphi_1(z)$ and $E \exp(zY_2) = \varphi_2(z)$ are entire functions of the complex variable z. Further,

$$\varphi_1(z)\varphi_2(z) = \varphi(z), \tag{2}$$

so that $\varphi_1(z)$ and $\varphi_2(z)$ have no zeros. Put $\varphi_1(z) = \exp g(z)$, where $g(z)$ is real on the real axis.

From the properties of characteristic functions and (2) we derive

$$|\varphi(x+iy) \exp(\pm g(x+iy))| \leq \varphi(x) \exp(\pm g(x)), \tag{3}$$

where the inequality is valid for both signs. Setting

$$g(z) = u(x, y) + iv(x, y), \tag{4}$$

we derive the fundamental inequality

$$|u(x, y) - u(x, 0)| \leq \gamma y^2 + 2\lambda e^x \sin^2 \frac{y}{2} \tag{5}$$

and its very important special case

$$|u(x, 2\pi m) - u(x, 0)| \leq 4\gamma^2 \pi^2 m^2 \tag{6}$$

for integral m.

From (5) and the basic properties of characteristic functions one obtains the important estimate

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* Incidentally, instead of them one may use the “basic functions” of L. Schwartz, used in the theory of generalized functions.

$$u(x,y) = O(x^2 + y^2 + e^x). \tag{7}$$

With the aid of the Poisson integral (see, for example, (3)) one estimates $v(x,y)$, after which the estimate

$$\begin{aligned} &|g(z)| = O(|z|^3 + 1) && \text{for } x \leq 0; \\ &|g(z)| = O(\exp|z|(1+\varepsilon)) && \text{for } x > 0 \text{ and any } \varepsilon > 0 \end{aligned}$$

is obtained.

Consideration of the two functions

$$G_{\pm}(z) = \gamma z^2 + \lambda e^z \pm g(z)$$

shows that at least one of them has exponential type 1 (4). Denote it by $G(z)$. Then

$$G(z) = O(|z|^3 + 1) \quad \text{for } x \leq 0. \tag{8}$$

Next, $G(z)$ is represented with the aid of the well-known Paley-Wiener theorem (5). This theorem concerns functions of exponential type belonging to the space L_2 on the imaginary axis. From (8) it is clear that the auxiliary function

$$H(z) = z^{-4}\left(G(z) - G(0) - G'(0)z - G''(0)\frac{z^2}{2} - G'''(0)\frac{z^3}{6}\right) \tag{9}$$

will be entire of exponential type 1 and will belong to L_2 on the imaginary axis. By the Paley-Wiener theorem we therefore obtain

$$H(z) = \int_{-1}^{1} e^{zt}\phi(t)\,dt, \tag{10}$$

where $\phi(t)$ is square-integrable $\big(\phi(t) \in L_2(-1,1)\big)$ and real. The integral in (10), as well as all integrals used below, is understood in the Lebesgue sense. From (10) we find

$$G_0(z) = G(z) - P(z) = z^4 \int_{-1}^{1} e^{zt}\phi(t)\,dt, \tag{11}$$

where $P(z)$ is the cubic polynomial appearing in the numerator of (9).

The further arguments are aimed at proving that $\phi(t) \approx 0$ for $t < 0$ and $\phi(t) \approx Q_3(t)$ for $t > 0$, where $Q_3(t)$ is a cubic polynomial ($\approx$ is the sign of equivalence).

The equality $\phi(t) \approx 0$ $(t < 0)$ is obtained by a not very complicated calculation based on (7).

The proof would be greatly simplified if it were possible to integrate (11) by parts so as to remove the troublesome factor z^4 at the cost of the appearance of simple nonintegral terms. However, $\phi(t)$ is merely a summable function, and integration of (11) by parts apparently requires the use of generalized functions. Since some further quantitative estimates are still needed, this route had to be abandoned in favor of a considerable detour.

Let $\{\gamma_m\}$ be a set of real numbers depending on the parameter x $(z = x + iy)$, such that

$$|\gamma_m| < \frac{A_0(x)^5}{(m+1)^{20}} \tag{12}$$

(in what follows c_i are positive constants; $A_0(x)$ is a positive function of x). We introduce into consideration the absolutely convergent even Fourier series:

$$\Lambda(t,x) = \sum_{m=0}^{\infty} \gamma_m \cos 2\pi m t. \tag{13}$$

Let $G_0(z) = u_0(x,y) + iv_0(x,y)$; consider the operation

$$\Phi(\Lambda, u_0) = \sum_{m=0}^{\infty} \gamma_m u_0(x, 2\pi m). \tag{14}$$

If the γ_m are subject to the very important condition

$$\sum_{m=0}^{\infty} \gamma_m = 0, \tag{15}$$

then the series

$$\Phi(\Lambda, u_0) = \sum_{m=0}^{\infty} \gamma_m \left[u_0(x, 2\pi m) - u_0(x, 0)\right] \tag{16}$$

will converge by virtue of (6), and for $|\Phi(\Lambda, u_0)|$ one can obtain a sufficiently good upper estimate. Further, from (11) we obtain

$$\Phi(\Lambda, u_0) = \int_0^1 \frac{\partial^4}{\partial t^4}(e^{xt}\Lambda(t,x))\phi(t)\,dt \tag{17}$$

(here it has been taken into account that $\phi(t) \approx 0$ for $t < 0$).

(17), (16), and (6) are applied in order to prove the main fact: for a given sufficiently small h,

$$\Delta^4\phi(t) = \phi(t+4h) - 4\phi(t+3h) + 6\phi(t+2h) - 4\phi(t+h) + \phi(t) = 0 \qquad (18)$$

for almost all $t \in (0,1)$. For this purpose one first considers values $t_0 \in (1/2, 1)$ that are Lebesgue points of $\Delta^4\phi(t)$ for the given h (see (6), p. 224); almost all points at which $\Delta^4\phi(t)$ is defined will be Lebesgue points. We give a concrete construction of the required $\Lambda(t,x)$.

First introduce a continuous positive $W(\xi)$ of the "smooth peak" type; the most convenient here are "I. M. Vinogradov cups" ([7], pp. 27-29) or suitable "basic functions" ([8], p. 10); $0 \le W(\xi) \le 1$;

$W(\xi) = 0$ $(|\xi| > \xi_0)$; ξ_0 is small, i.e. $(-\xi_0, \xi_0)$ is the support of $W(\xi)$. We form

$$\begin{gathered} U(v,x,t_0) = W(v-t_0-4h) - 4e^{xh}W(v-t_0-3h)+ \\ +6e^{2xh}W(v-t_0-2h) - 4e^{3xh}W(v-t_0-h) + e^{4xh}W(v-t_0). \end{gathered} \qquad (19)$$

Next we set

$$\Gamma(t,x) = \frac{1}{6}\int_0^t e^{x(v-t)}(t-v)^3 U(v,x,t_0)\,dv, \qquad (20)$$

so that

$$\frac{\partial^4}{\partial t^4} e^{xt}\Gamma(t,x) = e^{xt}U(t,x,t_0). \qquad (21)$$

At the same time it is very important that

$$\Gamma(0,x) = \Gamma(1,x) = 0, \qquad (22)$$

so that (15) is satisfied. Now $\Lambda(t,x)$ is constructed in the form

$$\Lambda(t,x) = \Gamma(t,x) + \Gamma(1-t,x), \qquad (23)$$

which ensures evenness in t for all x. A suitable choice of $\xi_0 = \xi_0(x, t_0)$, the estimate of $|\Phi(\Lambda, u_0)|$ by means of (16) from above, and the use of the fact that $t_0 \in (1/2, 1)$ is a Lebesgue point of $\Delta^4\phi(t)$, as $x \to \infty$, lead to a contradiction of the assumption that $\Delta^4\phi(t_0) \ne 0$, so that $\Delta^4\phi(t) = 0$ almost everywhere for $t_0 \in (1/2, 1)$. Hence, on the basis of simple facts from the theory of summable functions, it follows that $\phi(t) \simeq Q_3(t)$ (a cubic polynomial) for $t \in (1/2, 1)$. In the integral (11), $\phi(t)$ is replaced by $Q_3(t)$ on the segment $[1/2, 1]$, and after integration everything is transferred to the left-hand side or introduced under the integral sign on the segment $[0, 1/2]$. After this it is already considerably simpler to prove that $\phi(t) \simeq Q_3(t)$ also on the segment $[0, 1/2]$. Then $G(z)$ is directly brought to the form

$$G(z) = P(z) + c_1 e^z + e^{z/2}(a_0 z^3 + a_1 z^2 + a_2 z + a_3) + a_4,$$

from which the form of $g(z)$ is easily found, and simple arguments lead to the main theorem.

By the method described, it is possible to investigate decompositions of fairly broad classes of laws.

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Note: Figure translations are in progress. See original paper for figures.

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