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Abstract

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ON TOPOLOGICAL SPACES WHOSE WEIGHT IS GREATER THAN THEIR CARDINALITY

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P. S. Urysohn constructed an example of a T_5-space (with a single non-isolated point) of cardinality $\aleph_0$ and weight $> \aleph_0$ ((1), p. 206). Subsequently M. Bebutov and V. Schneider constructed an example of a T_2- (but not T_3!) space of cardinality $\aleph_0$, the weight at each point of which is greater than the cardinality (2). If one takes an R-minimal Hewitt–Katětov space (3), obtained, for example, from the space of rational numbers, then one easily obtains an existence theorem for countable T_5-spaces (cf. 1.2 and 1.3 of (6)) whose weight at each point is greater than the cardinality. However, this still does not give an individual example of such a space, since (and this is acknowledged by Katětov (3)) it is not possible to construct an individual R-minimal space. Moreover, an existence theorem for analogous uncountable spaces cannot be proved by means of R-minimal spaces.

The present paper is devoted to the following questions (we use the terminology and notation of (6); in particular, $\aleph_\alpha$ denotes a regular, $\aleph_\lambda$ an irregular, and $\mathfrak{m}$ an arbitrary cardinal number): a) enumeration of the cardinality of all T_5-spaces of cardinality $\mathfrak{m}$ and weight $> \mathfrak{m}$; b) construction of individual examples of such spaces; c) construction of an individual T_5-(T_3-)space of regular (irregular) cardinality $\aleph_\alpha$ ($\aleph_\lambda$), the weight at each point of which is greater than the cardinality; d) construction of a completely simple countable T_2-(T_5-)space with uncountable weight at each (at one) point, and also of a countable connected T_2-space with uncountable weight at each point; e) a negative solution of the problem of $[ab]$-compactness (for irregular α); the solution of it undertaken in (4) contains, as P. S. Aleksandrov kindly informed me, a gap (the inevitability of which is now also becoming clear).

Since, in view of (6), the set of all T_5-spaces of cardinality $\mathfrak{m}$ has cardinality $2^{2^{\mathfrak{m}}}$, while the set of all T-spaces of cardinality and weight $\leq \mathfrak{m}$ has cardinality $2^{\mathfrak{m}}$, it follows that the cardinality of the set of all T_5-spaces of cardinality $\mathfrak{m}$ and weight $> \mathfrak{m}$ is $2^{2^{\mathfrak{m}}}$—greater than the cardinality of all spaces of cardinality and weight $\leq \mathfrak{m}$, and a) is accomplished.

Let $\mathfrak{S}_\lambda = \sum_{\nu<\lambda}^{(A)} \Omega_{\nu+1}$ and $\mathfrak{P}_\alpha = \Omega_\alpha \times \Omega_\alpha$ (6).

Lemma 1. $\mathfrak{P}_\alpha$ is a $T_1^\alpha \overline{T}_2^{+}$-($\aleph_\alpha$)-space, whose local weight is $> \aleph_\alpha$ (cf. 2.2 of

([6])), and $\mathfrak{S}_\lambda$ is a $T_1\overline{T}_2^+$-$(\aleph_\lambda)$-space, whose local weight is $> \aleph_\lambda$.

The first part of the lemma is proved by contradiction with the help of 1.2 and 2.8 of ([6]) and the fact that in every set $P \subseteq \mathfrak{P}_\alpha$ there is contained a set equipotent to P and closed in itself.

We shall prove the second part of the lemma. It is clear that for an arbitrary point $x_0 \in \mathfrak{S}_\lambda$ the pseudo-weight $\psi(x_0) = \aleph_\lambda$ (cf. 2.7 of ([6])), so that the weight $\chi(x_0) \geq \psi(x_0) =$

$= \aleph_\lambda$, and it remains for us to show that $\chi(x_0) \geq \aleph_\lambda$, which we shall do by contradiction. Let $\chi(x_0) = \aleph_\lambda$. Thus, let $\mathfrak{G}(x_0) = \{G^\xi(x_0)\}_{\xi<\omega_\lambda}$ be an open base $\mathfrak{S}_\lambda$ at x_0, where $G_\xi(x_0) = \bigcap_{\nu<\lambda} G^\xi_{\nu+1}$ and $G^\xi_{\nu+1}$ $(\xi < \omega_\lambda)$ are nonempty and open in $\Omega_{\nu+1}$, with $x_0 \in G^\xi_{\nu_0+1}$ $(\xi < \omega_\lambda)$. Put

$$G_{\nu+1} = \left(\bigcap_{\xi\leq\omega_\nu} G^\xi_{\nu+1}\right) \setminus y_{\nu+1},$$

where $y_{\nu+1} \neq x_0$ and $y_{\nu+1}$ belongs to the indicated intersection. Clearly,

$$G(x_0) = \bigcup_{\nu<\lambda} G_{\nu+1}$$

is a neighborhood of x_0 in $\mathfrak{S}_\lambda$. We take in the base $\mathfrak{G}(x_0)$ a neighborhood $G^{\xi_0}(x_0) \subset G(x_0)$; then $G^{\xi_0}_{\nu+1} \subset G_{\nu+1}$ for every $\nu < \lambda$, and from the definition of $G_{\nu+1}$ we have $\xi_0 > \omega_\nu$ for every $\nu < \lambda$, i.e. $\xi_0 \geq \omega_\lambda$. But $\xi_0 < \omega_\lambda$, and we obtain the required contradiction.

Definition. Let $\mathfrak{M}$ be a T-space and $x_0 \in \mathfrak{M}$. By $x_0(\mathfrak{M})$ we shall agree to denote the space obtained from $\mathfrak{M}$ by weakening the topology by declaring all points of $\mathfrak{M}$, except x_0, isolated.

Lemma 2. If $\mathfrak{M}$ is a T_1-space and $x_0 \in \mathfrak{M}$, then $x_0(\mathfrak{M})$ is a T_5-space, and the weights of $\mathfrak{M}$ and $x_0(\mathfrak{M})$ at x_0 coincide.

Now the assertion required for b) follows at once from the lemmas given above. Let

$$M_\omega = \bigcup_{0\leq n<\omega} M_n,$$

where the M_n are pairwise disjoint sets, each of cardinality $\mathfrak{m}$, and let φ be a one-to-one mapping of $M_\omega \setminus M_0$ onto M_ω, subject to the conditions: 1) $\varphi(M_{n+1}) = M_n$ $(n = 0, 1, 2, ...)$, and 2) $\varphi^{-1}(x_n) = M^{x_n}_{n+1}$ has cardinality $\mathfrak{m}$ $(x_n \in M_n)$; let $\mathfrak{M}^{x_n}_{n+1}$ be some $T_1T_2^1$-space defined on $M^{x_n}_{n+1}$. We shall then agree to associate with the set M_ω the space $\mathfrak{M}_\omega$, defined on M_ω by the neighborhoods:

$$U(x_n) = x_n \cup G^{x_n}_{n+1} \cup \left\{\bigcup_{i<\omega} \varphi^{-i}\left(G^{x_n}_{n+1}\right)\right\},$$

where $x_n \in M_n$, $G_{n+1}^{x_n}$ is an arbitrary nonempty open set in $\mathfrak{M}_{n+1}^{x_n}$, and φ^{-i} are defined inductively: $\varphi^{-2} = \varphi^{-1}\varphi^{-1}$, etc. $\mathfrak{M}_\omega$ is zero-dimensional and therefore is always a T_3-space.

Lemma 3. The weight of $\mathfrak{M}_\omega$ at x_n is not less than the weight of the space $\mathfrak{M}_{n+1}^{x_n}$.

Proof. We shall regard $M_{n+1}^{x_n} \cup x_n$ as a subspace in $\mathfrak{M}_\omega$; in this subspace all points of $M_{n+1}^{x_n}$ are isolated, and it is homeomorphic to the space $x_n(\mathfrak{M}_{n+1}^{x_n} \cup x_n)$, where in the parentheses stands the space obtained from $\mathfrak{M}_{n+1}^{x_n}$ by adjoining the point x_n in the manner 2,3 of ([6]). In view of 2,3 of ([6]) and Lemma 2, the weight of $M_{n+1}^{x_n} \cup x_n \subset \mathfrak{M}_\omega$ at x_n is equal to the weight of $\mathfrak{M}_{n+1}^{x_n}$, and hence the weight of $\mathfrak{M}_\omega$ at x_n is not less than the weight of $\mathfrak{M}_{n+1}^{x_n}$, as was required to prove.

Putting now identically $\mathfrak{M}_{n+1}^{x_n} \equiv \mathfrak{S}_\lambda$, or $\equiv \mathfrak{P}_\alpha$, we obtain a T_3-space $\mathfrak{M}_\omega$ whose weight at each point is greater than its cardinality. If $\mathfrak{m} = \aleph_\alpha$ and $\mathfrak{M}_{n+1}^{x_n} \equiv \mathfrak{P}_\alpha$, then, since $\mathfrak{M}_\omega$ is then an $\aleph_{\alpha+1}$-bicompact T_3-space, in view of 1,5 of ([6]) $\mathfrak{M}_\omega$ is a T_4-space, and since all properties used for the proof of the axiom T_4 are hereditary, $\mathfrak{M}_\omega$ is a T_5-space, and c) is accomplished.

Remark. If $\mathfrak{m} = \aleph_0$ and $\mathfrak{M}_{n+1}^{x_n} \equiv \Omega_0$ ([6]), then $\mathfrak{M}_\omega$ is homeomorphic to the space of rational numbers R, so that our space $\mathfrak{M}_\omega$ in the general case is obtained by weakening the topology in R (since every countable T_1-space is obtained by weakening the topology of Ω_0).

Let (as above) R be the space of rational numbers with the usual topology; by $\widetilde{R}$ denote the space obtained from R by weakening the topology according to the rule: as closed sets in $\widetilde{R}$ one regards all sets of the form $F \cup N$, where F is closed and N is nowhere dense in R.

It is quite easy to see that $\widetilde{R}$ is an F_2-space with uncountable weight at each point; the space $x_0(\widetilde{R})$, for any $x_0 \in \widetilde{R}$, is, by elementary lemma 2, a countable T_5-space with uncountable weight, simpler than the space of P. S. Urysohn [1]. Further, if at the vertices of the lower bases of the triangles that serve Bing [5] for constructing a countable connected T_2-space, instead of the sets $\delta_1 \cap R$ and $\delta_2 \cap R$ (δ_1 and δ_2 are intervals on the abscissa axis) one takes the sets $\delta_1 \cap (R \setminus N_1)$ and $\delta_2 \cap (R \setminus N_2)$, where N_1 and N_2 are nowhere dense in R, then we obtain a countable connected T_2-space with uncountable weight at every point, and d) is realized.

Finally, to realize e), we shall show that an isolated space $\mathfrak{E}_\lambda$ of irregular cardinality $\aleph_\nu$ is an $[\aleph_\lambda, \aleph_{\lambda+1}]$-compact space for which the property $B^0_{\aleph_\lambda, \aleph_{\lambda+1}}$ [2] is not fulfilled, even if for it one considers covers not designated, i.e. with pairwise distinct elements. Let f be a one-to-one mapping of $\mathfrak{E}_\lambda$ onto $\mathfrak{S}_\lambda$; take in $\mathfrak{E}_\lambda$ a cover $\mathfrak{B}_{\lambda+1}$ of cardinality $\aleph_{\lambda+1}$, into which enter all one-point and some other sets that are carried by means of f into proper closed subsets of $\mathfrak{S}_\lambda$ (they may be regarded as pairwise distinct, since the weight of $\mathfrak{S}_\lambda$ is greater than $\aleph_\lambda$). Since $\mathfrak{S}_\lambda$ is a $\overline{T}_2(\aleph_\lambda)$-space, no subcover of cardinality $< \aleph_\lambda$ can be selected from $\mathfrak{B}_{\lambda+1}$, as was required. But the required $\mathfrak{B}_{\lambda+1}$ can be constructed still more simply: it is enough to represent $\mathfrak{E}_\lambda$ as the union of disjoint sets $\mathfrak{E}_\lambda = \mathfrak{E}_\lambda^0 \cup \mathfrak{E}_\lambda^1$, each of cardinality $\aleph_\lambda$; then as $\mathfrak{B}_{\lambda+1}$ one may take the union $\mathfrak{E}_\lambda^0 \cup \widetilde{\mathfrak{E}}_\lambda^1$, where $\widetilde{\mathfrak{E}}_\lambda^1$ is a cover of $\mathfrak{E}_\lambda^1$ of cardinality $\aleph_{\lambda+1}$.

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Note: Figure translations are in progress. See original paper for figures.