ON UNIFORMLY BEST APPROXIMATION OF POLYNOMIALS

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Abstract

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ON UNIFORMLY BEST APPROXIMATION OF POLYNOMIALS

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Let us consider the special case of the Chebyshev problem on the best approximation of $f(x)$, continuous on $[0,1]$, by polynomials of a prescribed degree. Let $f(x) = P_n(x)$; for $m < n$ one seeks $P_m(x)$ for which $\max_{[0,1]} |P_n(x) - P_m(x)|$ has the least value L. As is known, such a polynomial $P_m(x)$ is unique, and the difference $P_n(x) - P_m(x)$ attains $\pm L$ at not fewer than $(m+2)$ points on $[0,1]$, with successive alternation of sign. Setting $P_n(x) - P_m(x) = Y_n(x)$, we obtain the following formulation of the problem: among polynomials of degree n with prescribed $n-m$ leading coefficients, find the one which on $[0,1]$ deviates least from zero, and the deviation L itself. Thus the number s of deviation points (nodes) of the polynomial $Q_n(x) = Y_n(x)/L$ is subject to the condition $s \geq m+2$.

The problem posed is completely solved if $Q_n(x)$ is a polynomial of class II, i.e., if the number of its nodes $s > n/2 + 1$ for $\max_{[0,1]} |Q_n| = 1$. For this it is sufficient that $m + 2 > n/2 + 1$. Hence:

$$m < n < 2m + 2. \tag{1}$$

The results obtained by us for polynomials of class II make it possible to regard these polynomials as known; we shall present them.

The characteristics of polynomials of class II are: n—the degree, s—the number of nodes $(\sigma_i)_1^s$ on $[0,1]$, p—the number of repetitions of the sign of the polynomial at the boundaries of the intervals (σ_i, σ_{i+1}); they form the passport of the polynomial $[n, s, p]$; by q we denote the number of alternations of signs of the polynomial, so that $p + q = s - 1$ ([1]).

For example, the passport $[n, n+1, 0]$ characterizes the polynomials $\pm T_n(x) = \pm \cos n \arccos(2x-1)$ and only them; here $q = n$.

The totality of the nodes of a polynomial, to each of which is assigned the sign taken by the polynomial at this point, is called its distribution: $(\underset{\pm}{\sigma_i})_1^s$.

The existence of polynomials of any passport, their properties, and their analytic construction are obtained by means of the simplest linear functionals. Each such

$Q_n(x)$ is an extremal polynomial of a certain segment-functional F, namely: if we prescribe the parameters $(\mu_i)_0^n$ and set $F(x^i) = \mu_i$ $(i = 0, 1, \dots, n)$, $F(P_n) = \sum_0^n p_i \mu_i$, then there exists an extremal polynomial $Q_n(x)$ for which $F(Q_n) = +N$, where N is the norm of F; under the condition $s > n/2 + 1$ the extremal polynomial is unique ([2]).

Theorem 1. *All polynomials of a given passport* $[n, s, p]$ *form a family depending on* l *independent variable parameters, where*

$$l + s = n + 1. \tag{2}$$

For the problem posed above, consider the family of polynomials of passport $[n, s, 0]$, under the assumption that all polynomials of higher passports (i.e., with a larger number of nodes) have already been studied.

Theorem 2. The segment-functional

$$(\mu_i)_0^n = 0_0,\ 0_1, \dots, 0_{n-l+1},\ 1_{n-l},\ \vartheta_1,\ \vartheta_2, \dots, \vartheta_l \tag{*}$$

completely determines all polynomials of passport $[n, s, 0]$ (up to sign)

$$Q_n(x, \vartheta_1, \vartheta_2, \dots, \vartheta_l),$$

when the parameters $(\vartheta_i)_1^l$ vary in some bounded l-dimensional domain M_l. At each interior point of M_l the segment determines one extremal polynomial of passport $[n, s, 0]$; conversely, any given polynomial of such a passport is, up to sign, an extremal polynomial $(*)$ at one and only one interior point of M_l.

Theorem 3. To points $(\vartheta_i)_1^l$ of the l-dimensional space lying outside or on the boundary of M_l, the segment $(*)$ corresponds, as extremal polynomials, in a unique way (but not one-to-one) all (and only) polynomials of higher passports for which the number of alternations $q > s - 1$.

(The last assertion is obvious, since when nodes are added to an existing distribution $(\sigma_i)_1^s$, the number of alternations cannot decrease.)

Theorem 4. In the family of polynomials

$$Q_n(x, \vartheta_1, \vartheta_2, \dots, \vartheta_l),$$

defined by the segment $(*)$ in the domain M_l, the deformation parameters (ϑ_i) may be replaced by other parameters completely equivalent to them, (θ_i)—the highest coefficients of the polynomial, namely:

$$Q_n(x, \vartheta_1, \dots, \vartheta_l) = \theta_l x^n + \theta_{l-1} x^{n-1} + \dots + \theta_1 x^{n-l+1} + y_{n-l}(\theta_1, \dots, \theta_l) x^{n-l} + \cdots.$$

The one-to-one correspondence between the old and new parameters is expressed by the formulas

$$\vartheta_1 = -\frac{\partial y_{n-l}}{\partial \theta_1}, \dots, \vartheta_l = -\frac{\partial y_{n-l}}{\partial \theta_l}. \tag{3}$$

Let us note that Theorem 4 extends without change of formulation to polynomials of any other passport $[n, s, p]$; only in the case $p > 0$ will the defining functional, which also contains l variable parameters, be of a form different from $(*)$, and formulas (3) will likewise be replaced by others (see, for example, (3) for the passport $[n, n, 1]$).

We shall call the resolvent of a polynomial of class II $Q_n(x)$ with distribution $(\sigma_i)_1^s$ the polynomial

$$R_s(x) = \prod_1^s (x - \sigma_i) = x^s + \alpha_{s-1} x^{s-1} + \cdots.$$

In the case where the entire family of polynomials of a given passport

$$Q_n(x, \theta_1, \theta_2, \dots, \theta_l)$$

is considered (see Theorem 4), we have

$$\alpha_i = \alpha_i(\theta_1, \theta_2, \dots, \theta_l).$$

Polynomials of class II, distributed by passports, reveal a deep unity of analytic structure.

Theorem 5. For polynomials of any passport $[n, s, p]$

$$Q_n(x_1, \theta_1, \dots, \theta_l) = \theta_l x^n + \theta_{l-1} x^{n-1} + \cdots + \theta_1 x^{n-l+1} + y_{n-l} x^{n-l} + \cdots$$

the following differential relations hold:

$$\frac{\partial Q_n}{\partial \theta_i} = \varphi_{i-1}(x) R_s(x), \qquad i = 1, 2, \dots, l, \tag{4}$$

i.e., the derivative of a class-II polynomial with respect to each of its deformation parameters θ_i is always a multiple of the resolvent of the polynomial.

The coefficients of the polynomial $\varphi_{i-1}(x)$ are easily expressed in terms of derivatives of the coefficients (y_i) from the condition that certain coefficients of the polynomial $\partial Q_n / \partial \theta_i$ vanish.

Theorem 5 gives a method for the analytic construction of polynomials of class II as integrals of a system of ordinary differential equations; for this it is quite sufficient to use the first equation (4) with $i = 1$, i.e.

$$\frac{\partial Q_n}{\partial \vartheta_1} = R_s(x). \tag{4'}$$

In this equation the resolvent is eliminated by means of the easily verified relation

$$x(x-1)\frac{\partial Q_n}{\partial x} = n\vartheta_1 R_s(x)\psi_l(x). \tag{5}$$

Here $\psi_l(x) = \prod_1^l (x-\lambda_i)$, where (λ_i) are the roots of $\partial Q_n/\partial x$ which are not among the nodes of Q_n. We note that formula (5) is written for the case when the nodes (σ_i) include 0 and 1; all other cases reduce to this one (for the particular case, see (4)).

After the elimination of $R_s(x)$, equation (4) decomposes into a system of n equations, differential with respect to $s-1$ unknown functions $y_{s-1}, \dots, y_1$ ($y_0 = \pm 1$) and algebraic with respect to l unknown coefficients of $\psi_l(x)$.

Additional conditions for determining the constants of integration are obtained from the analytic form of the polynomial $Q_n(x)$ on the boundary of the domain M_l, where it becomes a polynomial of a higher passport. Thus, for example[5], in order to construct polynomials of passport $[n, n-1, 0]$ (the polynomials of N. I. Akhiezer), one must regard as already known the polynomials of passport $[n, n, 0]$ (i.e. the polynomials of E. I. Zolotarev).

Considering all polynomials of class II $Q_n(x)$ as known, let us return to the problem posed at the beginning of this note. Suppose that the leading coefficients $Y_n(x)$, $l+1$ in number, are given; denote them by $b_n, b_{n-1}, \dots, b_{n-l}$. The number s of nodes of the polynomial $Q_n(x) = Y_n(x)/L$ is subject to the condition $s \geq n+1-l$; consequently, $Q_n(x)$ is determined by the functional (*) in the domain M_l, if $s = n+1-l$, and by the same functional outside and on the boundary of M_l, if $Q_n(x)$ has a larger number of nodes.

Theorem 6. Among the polynomials determined by the segment (*) over the entire unbounded domain of variation $(\vartheta_i)_1^l$, there is one and only one polynomial $Q_n(x)$ whose coefficients $q_n, q_{n-1}, \dots, q_{n-l}$ are proportional respectively to the numbers $b_n, b_{n-1}, \dots, b_{n-l}$; if $1/M$ denotes the common value of the ratios q_i/b_i, then $|M| = L$ and $Y_n(x) = |M| Q_n(x)$.

To find $Q_n(x)$ and M, in the family of polynomials $Q_n(x, \vartheta_1, \dots, \vartheta_l)$ determined by the segment (*), one must put $\vartheta_i = \lambda b_i$ $(i = n, n-1, \dots, n-l+1)$ and

$$y_{n-l}(\lambda b_{n-l+1}, \dots, \lambda b_n) = \lambda b_{n-l}. \tag{6}$$

From equation (6) the unique (according to Theorem 6) real root $\lambda = \lambda_0$ is determined. If it turns out that $\lambda_0 < 0$, then the found $Q_n(x)$ should be replaced by $-Q_n(x)$. The deviation is $L = 1/\lambda_0$.

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Note: Figure translations are in progress. See original paper for figures.