ON A QUESTION OF N. N. LUZIN

1957

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Abstract

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MATHEMATICS

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ON A QUESTION OF N. N. LUZIN

(Presented by Academician A. N. Kolmogorov, 3 XII 1956)

1. In the well-known list of problems of N. N. Luzin (^{1}), under number 28 the following question is posed: "Let $f(x)$ be a continuous function; let $f_n(x)$ be the n-th approximation to it. At some points, of course, this approximation will be closest; at other points this approximation will not be very close. Will the measure of the set of these latter points be equal to 0? Of course, everything depends on the laws of approximation; Chebyshev or trigonometric approximations are of greatest interest."

Since, for any fixed value of n, in the case of best uniform approximations by polynomials of degree n it is not difficult to give an example of a function continuous on $[a, b]$ (depending on n) for which the measure of the set M_n of points where the corresponding maximal deviation is attained is arbitrarily close to $b - a$, then, as indicated in the commentary to this list, the question may be posed as follows: for every function $f(x)$ continuous on $[a, b]$, will not the measure of the set M_n be equal to zero for all sufficiently large n*? An example of a function $f(x)$, discontinuous on $[a, b]$, equal alternately to $+1$ and -1 on a certain infinite sequence of nonintersecting segments with endpoints converging to b, and linear on adjacent intervals, shows that for the class of functions continuous on a half-interval the answer to the question posed must be negative**.

In his lectures on constructive function theory, delivered at Dnepropetrovsk University, A. F. Timan expressed the supposition that the answer should also be negative for the class of functions $f(x)$ continuous on a closed interval, and that in this case the measure of the set M_n tends to zero as $n \to \infty$. The results of the present paper imply the validity of these suppositions.

2. Let $f(x)$ be a function continuous on $[a, b]$; $P_n(f; x)$ an algebraic polynomial of degree $\le n$ least deviating from $f(x)$ on this interval;

$$E_n(f) = \max_{a \le x \le b} |f(x) - P_n(f; x)|;$$

$M_n(f)$ the set of those points x of the given interval for which

$$E_n(f) = |f(x) - P_n(f; x)|.$$

Theorem 1. There exist a function $F(x)$ continuous on $[a, b]$ and an infinite sequence of natural numbers n_k $(k = 0, 1, \dots;\ n_0 < n_1 < \cdots)$ such that for every k

$$\operatorname{mes} M_{n_k}(F) > 0. \tag{1}$$

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* We quote the corresponding passage from the comments of N. K. Bari and D. E. Men' shov: "If we take some function $f(x)$ and consider for all n polynomials $T_n(x)$ that deviate least from it, then the question may be posed as follows: should not the measure of the set where $|f(x) - T_n(x)|$ attains its maximum Δ_n become equal to zero as soon as n becomes sufficiently large? This question, apparently, remains open."

** This observation belongs to A. F. Timan.

To construct the function $F(x)$, form a sequence of continuous functions $f_\nu(x)$ $(\nu = 0, 1, \dots)$ so that, for some infinite sequence of numbers n_ν $(n_0 = 0 < n_1 < n_2 < \cdots)$, the following conditions are satisfied:

1) $\operatorname{mes} M_{n_k}(f_\nu) > 0$ for $k = 0, 1, \dots, \nu$;

2) each set $M_{n_k}(f_\nu)$ $(k = 0, 1, 2, \dots, \nu)$ consists of a finite number of points and a finite number of segments;

3) the graph of the function $f_\nu(x)$ is "glued together" from a finite number of graphs of polynomials of degree not exceeding $n_\nu + 1$;

4) whatever the value of $\nu = 0, 1, 2, \dots$, for all $k \geq \nu$

$$P_{n_\nu}(f_k; x) \equiv P_{n_\nu}(f_\nu; x);$$

5) for all x on the interval $[a, b]$

$$|f_\nu(x) - f_{\nu+1}(x)| < 1/2^\nu \qquad (\nu = 0, 1, 2, \dots);$$

6) $P_{n_k}(f_\nu; x) \not\equiv P_{n_k+1}(f_\nu; x)$ $(k = 0, 1, 2, \dots;\ \nu = 0, 1, 2, \dots)$.

For this purpose, considering for definiteness the interval $[0, 1]$, set

$$f_0(x) = \begin{cases} 0, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 \leq x \leq 1. \end{cases}$$

Suppose the functions $f_0(x), f_1(x), \dots, f_\nu(x)$ and the indices $0 = n_0, n_1, \dots, n_\nu$ have already been chosen. Since the intersection $M_{n_k}(f_\nu) \cap M_{n_i}(f_\nu)$ consists of a finite number of points, each segment contained in $M_{n_k}(f_\nu)$ can be divided by the points of all the other sets $M_{n_i}(f_\nu)$ $(i \neq k;\ i, k \leq \nu)$ into only a finite number of parts. If λ'_k is the number of all parts of these segments and λ''_k is the number of all isolated points of the set $M_{n_k}(f_\nu)$, then choose $n_{\nu+1}$ so that

$$4\sum_{k=0}^{\nu}(\lambda'_k+\lambda''_k)\le n_{\nu+1}$$

and so that

$$P_{n_{\nu+1}}(f_\nu;x)\not\equiv P_{n_{\nu+1}+1}(f_\nu;x).$$

In the case when the intersection

$$M_{n_{\nu+1}}(f_\nu)\cap\left\{[0,1]-\sum_{k=0}^{\nu}M_{n_k}(f_\nu)\right\}$$

is nonempty and contains a point x_0, for $0<\delta<\varepsilon$ set

$$f_{\nu+1}(x)=\begin{cases}f_\nu(x), & x\in[0,1]-(x_0-\varepsilon,x_0+\varepsilon),\\ P_{n_{\nu+1}}(f_\nu;x)+f_\nu(x_0)-P_{n_{\nu+1}}(f_\nu;x_0), & x\in(x_0-\delta,x_0+\delta).\end{cases}$$

On the remaining two intervals we regard $f_{\nu+1}(x)$ as linear and continuous at the endpoints.

It can be shown that, for sufficiently small ε and δ, the function $f_{\nu+1}(x)$ satisfies all conditions 1)–6). If, however, the indicated intersection is empty, then there is an index $k_0\le\nu$ and some segment $[\xi,\eta]\subset M_{n_{k_0}}(f_\nu)$ with the properties:

1′) on $[\xi,\eta]$ there lies no point of the set

$$\sum_{\substack{k=0\\k\ne k_0}}^{\nu}M_{n_k}(f_\nu);$$

2′) on $[\xi,\eta]$ there is one and only one point $x_0\in M_{n_{\nu+1}}(f_\nu)$;

3′) the differences $f_\nu(x_0)-P_{n_{k_0}}(f_\nu;x_0)$ and $f_\nu(x_0)-P_{n_{\nu+1}}(f_\nu;x_0)$ have different signs.

In this case, for

$$0<\delta<\varepsilon<(\eta-\xi)/2^{\nu^2}$$

set

$$f_{\nu+1}(x)=\begin{cases}f_\nu(x), & x\in[0,1]-(x_0-\varepsilon,x_0+\varepsilon),\\ P_{n_{\nu+1}}(f_\nu;x)+f_\nu(x_0)-P_{n_{\nu+1}}(f_\nu;x_0), & x\in(x_0-\delta,x_0+\delta).\end{cases}$$

On the remaining two intervals we regard $f_{\nu+1}(x)$ as linear and continuous at the endpoints.

It can be shown that, for sufficiently small ε and δ, the function $f_{\nu+1}(x)$ satisfies conditions 1)–6).

The continuous function $F(x) = \lim_{\nu\to\infty} f_\nu(x)$, by virtue of 1)–6), satisfies condition (1), since

$$P_{n_\nu}(F;x) \equiv P_{n_\nu}(f_\nu;x) \quad (\nu = 0, 1, \dots),$$

and, consequently,

$$\operatorname{mes} M_{n_\nu}(F) \geq \operatorname{mes} \bigcap_{k=\nu}^{\infty} M_{n_\nu}(f_k) = \lim_{k\to\infty} \operatorname{mes} M_{n_\nu}(f_k) \geq$$

$$\geq \operatorname{mes} M_{n_\nu}(f_\nu) \prod_{k=\nu}^{\infty} \left(1 - \frac{1}{k^2}\right) > 0.$$

The theorem is proved.

3. Let now $s_n(f;x)$ be the partial Fourier sum of order n of a periodic function $f(x)$ with period 2π. Denote by $M_n(f)$ the set of points $x \in [0, 2\pi]$ for which the relation

$$|f(x) - s_n(f;x)| = \max_{0\leq t\leq 2\pi} |f(t) - s_n(f;t)|$$

holds. Then the following theorem is valid.

Theorem 2. *There exists a function $F(x)$, continuous on $[0, 2\pi]$, and an infinite subsequence of numbers n_k $(k = 0, 1, 2, \dots;\ n_0 < n_1 < n_2 < \cdots)$ such that, for every k,*

$$\operatorname{mes} M_{n_k}(F) > 0.$$

The theorem can be proved by a method in which the considerations of the preceding proof and some properties of Fourier series are used.

4. Let Q be some set of finite measure and let $\{f_n(x)\}_0^\infty$ be an arbitrary sequence of functions, bounded and measurable on Q, for which, for any $n = k$, the difference $f_k(x) - f_n(x)$ can take a constant value only on a subset of measure zero.

Theorem 3. *If $R_n = \sup_{x\in Q} |f_n(x)|$ and $M_n^{(\alpha)}$ is the set of those $x \in Q$ for which $|f_n(x)| = \alpha R_n$, then*

$$\lim_{n\to\infty} \operatorname{mes} M_n^{(\alpha)} = 0.$$

For the proof one should note that, for $k \neq n$,

$$\operatorname{mes}\{M_k^{(\alpha)} \cap M_n^{(\alpha)}\} = 0.$$

We indicate some important special cases of Theorem 2.

Corollary 1. Let $\varphi_0(x) \equiv 1,\ \varphi_1(x), \ldots$ be a sequence of continuous functions possessing the property that, for every n, the system $\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)$ is Chebyshev on $[a, b]$ (2). For any function $f(x)$ continuous on $[a, b]$, consider the polynomial

$$P_n(f; x) = \sum_{k=0}^{n} C_k^{(0)} \varphi_k(x) \quad (n = 0, 1, 2, \ldots)$$

of best approximation of order n. From Chebyshev' s theorem on the conditions for best approximation it is easy to conclude that, if $f(x)$ is not a finite linear combination of the functions $\varphi_k(x)$, the sequence $\{P_n(f; x)\}_0^\infty$ contains a subsequence of pairwise distinct polynomials $P_{k_n}(f; x)$ $(n = 0, 1, \ldots)$. Moreover, the difference $P_{k_n}(f; x) - P_{k_m}(f; x)$ $(n \neq m)$ takes a constant value only in co-

number of points. Setting $f_n(x) = f(x) - P_{h_n}(f; x)$ and using Theorem 2, we obtain that for every α

$$\lim_{n \to \infty} \operatorname{mes} M_{h_n}^{(\alpha)}(f) = 0.$$

It follows that $\operatorname{mes} M_n^{\alpha}(f) = 0$ as $n \to \infty$.

Corollary 2. Let $f(x)$ be a bounded measurable function of period 2π that is not a trigonometric polynomial; let $s_n(f; x)$ be its Fourier partial sums of order n; and let $M_n^{(\alpha)}$ be the set of those $x \in [0, 2\pi]$ for which

$$|f(x) - s_n(f; x)| = \alpha \sup_x |f(x) - s_n(f; x)|.$$

From Theorem 2 we conclude that

$$\lim_{n \to \infty} \operatorname{mes} M_n^{(\alpha)} = 0.$$

Corollary 3. Let $f(x)$ be a bounded and uniformly continuous function on the entire real axis, not an entire function of finite degree. For any $\sigma > 0$, consider the entire function $g_\sigma(f; x)$ of degree $\leq \sigma$ that deviates least from $f(x)$ on $(-\infty, \infty)$. If

$$A_\sigma(f) = \sup_{-\infty < x < \infty} |f(x) - g_\sigma(f; x)|$$

and $M_\sigma^{(\alpha)}$ is the set of those x for which

$$|f(x) - g_\sigma(f; x)| = \alpha A_\sigma(f),$$

then from Theorem 2 it follows that, for any finite interval $[a, b]$,

$$\lim_{\sigma \to \infty} \operatorname{mes}\{[a, b] \cap M_\sigma^{(\alpha)}\} = 0.$$

In conclusion, we consider it our duty to express our gratitude to Prof. A. F. Timan for suggesting the topic and for his guidance.

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named after the 300th anniversary of the reunification of Ukraine with Russia

Received
25 V 1956

REFERENCES

1. N. N. Luzin, *The Integral and Trigonometric Series*, 1951.
2. N. I. Akhiezer, *Lectures on Approximation Theory*, 1947.

Note: Figure translations are in progress. See original paper for figures.

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