ON UNSTEADY FLOW OF A VISCOUS FLUID BETWEEN PARALLEL POROUS WALLS

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Abstract

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ON UNSTEADY FLOW OF A VISCOUS FLUID BETWEEN PARALLEL POROUS WALLS

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In paper (1), Berman investigated the two-dimensional steady laminar flow of a viscous incompressible fluid between parallel porous walls, when the porosity of the walls is constant. If the x-axis is directed parallel to the walls, then the stream function found by Berman may be written in the form

$$\psi^*(x, y) = (1 - \lambda^0 x)\psi^0(y), \qquad \lambda^0 = 2v^0/h\bar{u}(0),$$

where h is the distance between the walls; v^0 is the permeability velocity; $\bar{u}(0)$ is the averaged value of the longitudinal velocity in the section $x = 0$. Assuming v^0 to be small, the author determines $\psi^0(y)$ approximately as the solution of a nonlinear differential equation of third order. Yuan (2) investigated $\psi^0(y)$ for large values of the Reynolds number. In the present article we consider unsteady flow with permeability depending only on time, taking as the initial state the flow characterized by the function ψ^*.

The stream function of the unsteady flow under the action of potential body forces satisfies the equation

$$\Delta(\nu\Delta\psi - \psi_t) = \psi_y\Delta\psi_x - \psi_x\Delta\psi_y. \tag{1}$$

On the walls of the channel $(y = 0,\ y = h)$, ψ_x must take the prescribed values $v^0(t)$ and $-v^0(t)$, while ψ_y must vanish; at the initial instant ψ coincides with ψ^*.

Represent ψ in the form

$$\psi = (1 - \lambda x)f(y, t), \qquad \lambda = 2v^0(t)/h\bar{u}(t), \tag{2}$$

where $\bar{u}(t)$ is the averaged value in the initial section $(x = 0)$. We shall regard λ as constant, which will be satisfied, for example, if at the end of the channel $(x = l)$ the longitudinal velocity is equal to zero; in this case, from the condition of incompressibility, we obtain

$$h\bar{u}(t) = 2lv^0(t), \qquad \lambda = 1/l.$$

In view of (1) and (2), for f we shall have the equation

$$\frac{\partial^2}{\partial y^2}\left(\nu f_{yy} - f_t\right) = \lambda\left(f f_{yyy} - f_y f_{yy}\right) \tag{3}$$

and the boundary conditions

$$f(0,t) = -v^0(t)/\lambda, \qquad f(h,t) = v^0(t)/\lambda;$$

$$f_y(0,t) = f_y(h,t) = 0, \qquad f(y,0) = \psi^0(y). \tag{4}$$

We represent the function f as the sum $F(y,t) + \varphi(y,t)$, where F satisfies equation (3) with zero right-hand side and the boundary conditions (4), while φ is the solution of the equation

$$\frac{\partial^2}{\partial y^2}\left(\nu \varphi_{yy} - \varphi_t\right) = \lambda\left(f f_{yyy} - f_y f_{yy}\right), \tag{5}$$

satisfying the corresponding homogeneous and boundary conditions.

The function F may be interpreted as the stream function of a one-dimensional flow in the region under consideration, with velocity F_y, for fixed walls and with a prescribed initial value equal to $\psi^0(y)$. In accordance with the boundary conditions (4), F may be represented in the form

$$F(y,t) = F^0(y,t) + \Phi(y,t) - y[\Phi(0,t) + F^0(0,t)] - -\Phi(0,t) - F^0(0,t) - v^0(t)/\lambda,$$

where

$$F^0(y,t) = \frac{1}{2\sqrt{\pi\nu t}}\int_0^h \psi^0(\eta)\exp\left[-\frac{(y-\eta)^2}{4\nu t}\right]d\eta;$$

Φ is the solution of the heat equation $\nu\Phi_{yy} - \Phi_t = 0$, vanishing at the initial instant and satisfying the boundary conditions

$$\Phi(h,t) - \Phi(0,t) - h\Phi_y(0,t) = \frac{2}{\lambda}v^0(t) + F^0(0,t) - F^0(h,t) + +hF^0_y(0,t) = F_1(t),$$

$$\Phi(h,t)-\Phi(0,t)-h\Phi_y(h,t)=\frac{2}{\lambda}v^0(t)+F^0(0,t)-F^0(h,t)++hF_y^0(h,t)=F_2(t).$$

Representing Φ in the form

$$\Phi(y,t)=\sqrt{\frac{\nu}{\pi}}\int_0^t\left\{\Phi_1(\tau)\exp\left[-\frac{y^2}{4\nu(t-\tau)}\right]+\right.$$

$$\left.+\Phi_2(\tau)\exp\left[-\frac{(y-h)^2}{4\nu(t-\tau)}\right]\right\}\frac{d\tau}{\sqrt{t-\tau}}, \tag{6}$$

by virtue of the boundary conditions, the unknown functions Φ_1 and Φ_2 are determined from the system of regular Volterra integral equations

$$\Phi_1(t)+\int_0^t[\Phi_1(\tau)K(t-\tau)+\Phi_2(\tau)L(t-\tau)]\,d\tau=F_1(t),$$

$$\Phi_2(t)+\int_0^t[\Phi_1(\tau)L(t-\tau)-\Phi_2(\tau)K(t-\tau)]\,d\tau=F_2(t), \tag{6a}$$

where

$$K(z)=\frac{1}{h}\sqrt{\frac{\nu}{\pi z}}\left[\exp\left(-\frac{h^2}{4\nu z}\right)-1\right],\quad L(z)=\frac{h}{2\sqrt{\pi\nu z^3}}\exp\left(-\frac{h^2}{4\nu z}\right)+K(z).$$

To determine φ we use the Green’ s function

$$G(y,\eta,t)=S(y,\eta,t)+g(y,\eta,t),$$

where

$$S(y,\eta,t)=\frac{1}{2\sqrt{\pi\nu t}}\int_0^y dy\int_0^y\exp\left[-\frac{(y-\eta)^2}{4\nu t}\right]dy; \tag{7}$$

$$g(y,\eta,t)=U(y,\eta,t)-yU_y(0,\eta,t)-U(0,\eta,t); \tag{8}$$

U is the solution of the heat-conduction equation, vanishing at the initial moment and satisfying, for $t>0,\ 0<\eta<h$, the boundary conditions:

$$U(h,\eta,t)-U(0,\eta,t)-hU_y(0,\eta,t)=-S(h,\eta,t),$$

$$U(h,\eta,t)-U(0,\eta,t)-hU_y(h,\eta,t)=hS_y(h,\eta,t)-S(h,\eta,t). \tag{8a}$$

It is therefore clear that the problem of determining g is again reduced to the solution of a system of two regular Volterra integral equations. Taking into account that

$$S_{yy}=-\frac{1}{2\sqrt{\pi\nu t}}\exp\left[-\frac{(y-\eta)^2}{4\nu t}\right], \tag{9}$$

as in the paper ([3]), it is not difficult to show by ordinary arguments that the following equality holds:

$$\varphi(y,t)=\lambda\int_0^t d\tau\int_0^h (ff_{\eta\eta\eta}-f_\eta f_{\eta\eta})G(y,\eta,t-\tau)\,d\eta;$$

denoting

$$H(\eta,\tau)=ff_{\eta\eta\eta}-f_\eta f_{\eta\eta},$$

we obtain

$$f(y,t)=F(y,t)+\lambda\int_0^t d\tau\int_0^h H(\eta,\tau)G(y,\eta,t-\tau)\,d\eta. \tag{10}$$

By virtue of formula (9), it is easy to see that the second term on the right-hand side of equality (10) may be differentiated with respect to y three times under the integral sign, and, thus, we shall have

$$\frac{\partial^n f}{\partial y^n}=\frac{\partial^n F}{\partial y^n}+\lambda\int_0^t d\tau\int_0^h H(\eta,\tau)\frac{\partial^n G}{\partial y^n}\,d\eta, \qquad n=1,2,3. \tag{10a}$$

Equalities (10), (10a) form a system of nonlinear integral equations for determining the unknown functions f, $\partial^n f/\partial y^n$, which may be found by successive approximations. For this purpose we represent the sought functions in the form of series

$$\frac{\partial^n f}{\partial y^n}=\sum_{k=0}^{\infty}\lambda^k\frac{\partial^n f_k}{\partial y^n}, \qquad n=0,1,2,3\left(\frac{\partial^0 f}{\partial y^0}=f\right). \tag{11}$$

To determine the terms of the series we have the following recurrence formulas:

$$\frac{\partial^n f_0}{\partial y^n} = \frac{\partial^n F}{\partial y^n}, \qquad \frac{\partial^n f_{k+1}}{\partial y^n} = \int_0^t d\tau \int_0^h \sum_{m=0}^{k} \left(f_m \frac{\partial^3 f_{k-m}}{\partial \eta^3} - \frac{\partial f_m}{\partial \eta} \frac{\partial^2 f_{k-m}}{\partial \eta^2} \right) \frac{\partial^n G}{\partial y^n}\, d\eta.$$

Convergence may be investigated by the method applied by Odkvist ([4]). For this purpose we note that it is not difficult to establish the validity of the equalities

$$\int_0^h \sqrt{t-\tau} \left| \frac{\partial^n G}{\partial y^n} \right| d\eta < c, \qquad n = 0, 1, 2, 3, \tag{12}$$

where c is a constant.

Let M be a constant such that

$$\left| \frac{\partial^n F}{\partial y^n} \right| < M, \qquad n = 0, 1, 2, 3; \tag{12a}$$

the majorant of the series (11) can be written in the form

$$A = \sum_{k=0}^{\infty} \lambda^k A_k, \qquad A_{k+1} = Mb\sqrt{t} \sum_{m=0}^{k} A_m A_{k-m},$$

where $b = 4c,\ A_0 = M$.

It is verified directly that it must be

$$A = A_0 + Mb\lambda\sqrt{t}\, A^2 = M\left(1 + b\lambda\sqrt{t}\, A^2\right),$$

whence it is seen that in a neighborhood of $\lambda = 0$ the value A exists for all λ smaller than the root of the discriminant $1 - 4M^2 b\lambda\sqrt{t}$. Hence we arrive at a sufficient condition for the existence of the expansions (11):

$$4M^2 b\lambda\sqrt{t} < 1.$$

If the last condition is fulfilled, the solution of the system (9), (10) is unique. This is easily proved by the usual arguments if we make use of the estimates (12) and (12a).

From equality (10) we can obtain the solution of the problem under consideration explicitly in various approximations. If the permeability velocity is so small in comparison with $u(t)$ that the quantity λ may be neglected, then we obtain the solution of the problem of one-dimensional flow between parallel walls. In the second approximation we shall neglect quantities of order λ^2; the solution of the problem is obtained by substituting into the right-hand side of equality (10) the value F instead of f.

The pressure p can be determined from the Navier–Stokes equations. Denoting by p_0 the pressure in the initial section, for the pressure difference over some segment x we obtain the formula

$$p - p_0 = -\frac{\rho}{2} B(t) x^2, \qquad B(t) = \nu f_{yyy} - f_{yt} + \lambda \left(f_y^2 - f f_{yy} \right),$$

where ρ is the density of the fluid.

If we compute the friction force per unit area of the boundary wall, we obtain

$$R = \mu(1 - \lambda x) f_{yy} = \mu F_{yy} - \mu \lambda x F_{yy} + (1 - \lambda x) \varphi_{yy}.$$

The first term on the right-hand side of the last equality is the friction force acting in one-dimensional flow in the absence of porosity.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.