SOME EFFECTIVE METHODS FOR SOLVING CHEBYSHEV'S PROBLEM ON BEST APPROXIMATION

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Abstract

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SOME EFFECTIVE METHODS FOR SOLVING CHEBYSHEV' S PROBLEM ON BEST APPROXIMATION

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Let D be a point set in the space of $2n+2$ dimensions R_{2n+2}, satisfying the following conditions: 1) if the point $P \equiv (x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n)$ belongs to D, then necessarily $a \leq x_0 < x_1 < \dots < x_n \leq b$, where $a < b$ are given real numbers; 2) for any fixed values $a \leq x_0^* < x_1^* < \dots < x_n^* \leq b$, all points of D of the form $(x_0^*, x_1^*, \dots, x_n^*, y_0, y_1, \dots, y_n)$ form in the $(n+1)$-dimensional subspace Y_{n+1} of the space R_{2n+2} a nonempty open parallelepiped D_{X^*} (not necessarily bounded).

Definition 1. We shall call a set K of functions $F(x)$, continuous on the segment $[a, b]$, an **interpolation class** (relative to the set D) if: 1) for any function $F(x) \in K$ and any values $a \leq x_0 < x_1 < \dots < x_n \leq b$, the point $(x_0, x_1, \dots, x_n, F(x_0), F(x_1), \dots, F(x_n))$ belongs to the set D; 2) whatever the point $P \equiv (x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_n) \in D$, there exists one and only one function $F(x) \in K$ for which

$$F(x_i) = y_i \qquad (i = 0, 1, \dots, n). \tag{1}$$

This definition includes, as special cases, the interpolation classes of functions considered in the works ([1–3]).

Definition 2. We shall agree to say that a function $f(x)$ of the argument x belongs to the class $C_D([a, b])$ if it is defined and continuous on the entire segment $[a, b]$ and, for any values $a \leq x_0 < x_1 < \dots < x_n \leq b$, the point $(x_0, x_1, \dots, x_n, f(x_0), f(x_1), \dots, f(x_n))$ belongs to the set D.

Let $f(x) \in C_D([a, b])$, and let S be some closed set of points of the segment $[a, b]$, containing no fewer than $n+2$ points. The problem consists in finding a function $F^*(x)$ of the class K that deviates least from $f(x)$ on S, i.e. such a function $F^*(x) \in K$ for which

$$\max_{x \in S} |f(x) - F^*(x)| = \inf_{F(x) \in K} \left\{ \max_{x \in S} |f(x) - F(x)| \right\} \equiv F[f; S]. \tag{2}$$

By the arguments of work ([2]), for problem (2) the fundamental theorems of the theory of Chebyshev approximation are easily established, in particular, the unique solvability of the indicated problem.

Definition 3. We shall call an interpolation class K **boundedly compact** if from every infinite bounded family of functions of the given class one can extract a subsequence converging uniformly to some function of the same class.

An example of an interpolation class that is not boundedly compact is the two-parameter family of functions $F(x; t_0, t_1) = t_0 e^{t_1 x}$ for $0 \leq x \leq 1$; $0 < t_0 < +\infty$; $-\infty < t_1 < +\infty$.

Definition 4. We shall call an interpolation class K **linear** if, together with any two of its functions $F_1(x)$ and $F_2(x)$, it also contains any linear combination $c_1 F_1(x) + c_2 F_2(x)$ of them for real c_1, c_2.

We proceed to methods for the actual solution of problem (2). The computational schemes proposed below are based on the solution of an auxiliary one-parameter problem, which is as follows: for a bounded and closed plane point set Q, it is required to determine the value of the real parameter $\lambda = \lambda^*$ from the condition

$$\max_{(u,v)\in Q} |v - \lambda u| \equiv L(\lambda) = \min. \tag{3}$$

A simple graph-analytic solution of the latter problem was proposed by E. Ya. Remez (4). However, this solution can also be found by a purely analytic method, by the formula

$$\lambda^* = \inf_{\substack{(u_1,v_1)\in Q_0 \\ u_1<0}} \left\{ \max_{\substack{(u_2,v_2)\in Q_0 \\ u_2\geq 0}} \frac{v_2 - v_1}{u_2 + |u_1|} \right\}, \tag{4}$$

where $Q_0 = Q \cup (-Q)$.

Let first the class K be linear, and let the exact solution $F_\varepsilon(x)$ of the analogous problem on a finite ε-net S_ε of the original set S be taken as the approximate solution of problem (2). To find the latter one may recommend the finite and monotone analogue of the second algorithm of E. Ya. Remez (5). Instead of the usual correction $\overline{F}(x)$, given by that algorithm to the approximating function $F(x)$, one takes each time a correction in the form $\overline{\lambda}\,\overline{F}(x)$, where the value $\lambda = \overline{\lambda}$ is, as a rule, the solution of the one-parameter problem (3) for the set Q_ε of pairs of numbers $(\overline{F}(x),\ f(x) - F(x))$, obtained for $x \in S_\varepsilon$. Correspondingly, the next set of $n+2$ points of S_ε is chosen for constructing the following approximation.

Suppose the function $F_\varepsilon(x)$ has already been found. If now $\varepsilon \to 0$, then always $E[f; S_\varepsilon] \to E[f; S]$, and in the case of bounded compactness of the class K also $F_\varepsilon(x) \to F^*(x)$ (uniformly with respect to x), where $F^*(x)$ is the function least deviating from $f(x)$ on S. In particular, if the class K coincides with the class H_n of all algebraic polynomials of degree not exceeding n, $S = [a, b]$, and $|f''(x)| \leq M_2$, then it turns out that

$$\max_{a \leq x \leq b} |f(x) - F_\varepsilon(x)| - E[f; S_\varepsilon] = O(\varepsilon^2). \tag{5}$$

This estimate, generally speaking, no longer permits an improvement of order, as is shown by the example of the function $f(x) = x^2$ for $n = 0$.

For finding an approximate solution of problem (2) directly on the entire segment $[a, b]$, one may propose a method that is a modification of the first algorithm of E. Ya. Remez (5) and its extension to the case of linear bounded compact interpolation classes. It consists in the following. We start from some "zero" approximation $F_0(x)$ of the function $f(x)$ on $[a, b]$ and set $\Delta_0(x) = f(x) - F_0(x)$. Suppose the function $\Delta_k(x)$ has already been constructed. Then it is possible to construct a function $F_{k+1}(x) \in K$ such that, in a sufficiently wide neighborhood of each point of absolute maximum of $|\Delta_k(x)|$ on $[a, b]$, the condition

$$\operatorname{sign} F_{k+1}(x) = \operatorname{sign} \Delta_k(x) \tag{6}$$

is satisfied. This is achieved by a proper choice of the zero function $F_{k+1}(x)$ on the segment $[a, b]$.

Let $\lambda = \lambda_{k+1}$ be the solution (even if only approximate) of the one-parameter problem (3) for the set Q of pairs of numbers $(F_{k+1}(x), \Delta_k(x))$, where x ranges ...

sweeps the whole segment $[a, b]$. Then we set $\Delta_{k+1}(x) = \Delta_k(x) - \lambda_{k+1} F_{k+1}(x)$, and so on. The process turns out to be convergent provided appropriate conditions are satisfied in the choice of the zeros of each of the functions $F_{k+1}(x)$ $(k = 0, 1, 2, \ldots)$.

We note that, for the application of the second algorithm of E. Ya. Remez[5], bounded compactness of the class K alone is sufficient. One need only take this algorithm in its "nonlinear" form, as it is presented in the paper[3].

For the final refinement of the magnitude of the deviation and of the alternation points, Newton' s method may also be applied.

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Note: Figure translations are in progress. See original paper for figures.