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Abstract

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PHYSICS

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ON THE INFLUENCE FUNCTION IN THE THEORY OF RADIANT ENERGY TRANSFER

(Presented by Academician M. A. Leontovich, 27 IX 1956)

1. Let $I(\nu, \vec{\omega}, \mathbf{r}, t)$ be the intensity of a beam of radiation of frequency ν with direction $\vec{\omega}$ at the point $\mathbf{r}$. It satisfies the transfer equation

$$\left\{\frac{1}{c}\frac{\partial}{\partial t} + (\vec{\omega}\nabla) + \alpha(\nu, \mathbf{r}) + \sigma(\nu, \mathbf{r})\right\} I(\nu, \vec{\omega}, \mathbf{r}, t) - \int_{\nu', \vec{\omega}'} p(\nu, \nu'; \cos\theta; \mathbf{r})\, \sigma(\nu', \mathbf{r})\, I(\nu', \vec{\omega}', \mathbf{r}, t) = j(\nu', \vec{\omega}', \mathbf{r}, t), \tag{1}$$

where c is the speed of light; $\alpha(\nu, \mathbf{r})$ is the coefficient of true absorption; $\sigma(\nu, \mathbf{r})$ is the scattering coefficient; $p(\nu, \nu', \cos\theta; \mathbf{r})$ is the probability of scattering of a light beam of frequency ν' through an angle θ with a change of frequency to ν; $j(\nu, \vec{\omega}, \mathbf{r}, t)$ is the source. If all these quantities are specified and do not depend on the intensity of the radiation, then equation (1) is linear. Therefore, if one introduces the influence function $G(\nu, \vec{\omega}, \mathbf{r}, t; \nu_0, \vec{\omega}_0, \mathbf{r}_0, t_0)$, i.e. the solution of the transfer equation with a point source

$$\left\{\frac{1}{c}\frac{\partial}{\partial t} + (\vec{\omega}\nabla) + \alpha(\nu, \mathbf{r}) + \sigma(\nu, \mathbf{r})\right\} G(\nu, \vec{\omega}, \mathbf{r}, t; \nu_0, \vec{\omega}_0, \mathbf{r}_0, t_0) -$$

$$- \int p(\nu, \nu'; \cos\theta; \mathbf{r})\, \sigma(\nu', \mathbf{r})\, G(\nu', \vec{\omega}', \mathbf{r}, t; \nu_0, \vec{\omega}, \mathbf{r}_0, t_0)\, d\nu'\, d\vec{\omega}' =$$

$$= \delta(\nu - \nu_0)\, \delta(\vec{\omega} - \vec{\omega}_0)\, \delta(\mathbf{r} - \mathbf{r}_0)\, \delta(t - t_0), \tag{2}$$

then any solution of equation (1) can be written in the form

$$I(\nu, \vec{\omega}, \mathbf{r}, t) = \int G(\nu, \vec{\omega}, \mathbf{r}, t; \nu', \vec{\omega}', \mathbf{r}', t')\, j(\nu', \vec{\omega}', \mathbf{r}', t')\, d\nu'\, d\vec{\omega}'\, d\mathbf{r}'\, dt'. \tag{3}$$

Equations (1), (2) must be supplemented by boundary conditions. As an initial condition it is natural to take

$$I(\nu, \vec{\omega}, \mathbf{r}, t) = 0 \quad \text{for } t = -\infty. \tag{4}$$

As for the boundary condition, we shall assume that the medium under consideration can be surrounded by such a convex surface S that it encloses all sources of radiation. Then on the boundary S the condition

$$I(\nu, \vec{\omega}, \mathbf{r}, t) = 0 \quad \text{for } \mathbf{r} \in S, \quad \mathbf{n}\vec{\omega} < 0, \tag{5}$$

will be satisfied, where $\mathbf{n}$ is the outward normal to S at the point $\mathbf{r}$.

Along with equation (1), let us introduce for consideration the adjoint equation for some function $I^*(\nu, \vec{\omega}, \mathbf{r}, t)$:

$$\left\{-\frac{1}{c}\frac{\partial}{\partial t} - (\vec{\omega}\nabla) + \alpha(\nu, \mathbf{r}) + \sigma(\nu, \mathbf{r})\right\} I^*(\nu, \vec{\omega}, \mathbf{r}, t) -$$

$$- \int I^*(\nu', \vec{\omega}', \mathbf{r}, t)\, \sigma(\nu, \mathbf{r})\, p(\nu', \nu; \cos\theta; \mathbf{r})\, d\nu'\, d\vec{\omega}' = i(\nu, \vec{\omega}, \mathbf{r}, t), \tag{6}$$

where we shall assume that I^* satisfies the boundary conditions adjoint to (4), (5):

$$\begin{aligned} I^*(\nu, \vec{\omega}, \mathbf{r}, t) &= 0 \qquad \text{for } t = +\infty; \\ I^*(\nu, \vec{\omega}, \mathbf{r}, t) &= 0 \qquad \text{for } \mathbf{r} \in S, \quad \mathbf{n}\vec{\omega} > 0. \end{aligned} \tag{7}$$

Let us write equation (1) in the form $LI = j$, where L is the integro-differential operator whose action is defined by formula (1). Equation (6) has the form $L^*I^* = i$, where L^* is the operator adjoint to L. The adjoint operator and the adjoint boundary conditions are uniquely determined by the requirement

$$\int (I^*LI - IL^*I^*)\, d\nu\, d\vec{\omega}\, d\mathbf{r}\, dt \equiv 0, \tag{8}$$

where the integration is carried out over all angles, frequencies, time, and the volume bounded by the surface S.

Substitute in (8) $I(\nu, \vec{\omega}, \mathbf{r}, t) = G(\nu, \vec{\omega}, \mathbf{r}, t; \nu_0, \vec{\omega}_0, \mathbf{r}_0, t_0)$ and $I^*(\nu, \vec{\omega}, \mathbf{r}, t) = G^*(\nu, \vec{\omega}, \mathbf{r}, t; \nu', \vec{\omega}', \mathbf{r}', t')$, where G^* is the influence function for equation (6). Taking into account the rule for integrating the δ-function, we obtain

$$G(\nu', \vec{\omega}', \mathbf{r}', t'; \nu_0, \vec{\omega}_0, \mathbf{r}_0, t_0) = G^*(\nu_0, \vec{\omega}_0, \mathbf{r}_0, t_0; \nu', \vec{\omega}', \mathbf{r}', t'). \tag{9}$$

This relation shows that the following reciprocity theorem holds: the influence function satisfies the adjoint equation with respect to the source point, with adjoint boundary conditions.

Hence the meaning of the adjoint condition (7) becomes clear: it means that if on the surface S there is a source sending radiation into the exterior space, then inside S there is no radiation. In addition, the reciprocity theorem makes it possible to reveal the physical meaning of the adjoint equation itself.

Indeed, any solution of equation (6) can, according to (9), be written in the form

$$I^*(\nu,\vec{\omega},\mathbf{r},t) = \int i(\nu',\vec{\omega}',\mathbf{r}',t')\, G(\nu',\vec{\omega}',\mathbf{r}',t';\nu,\vec{\omega},\mathbf{r},t)\, d\nu'\, d\vec{\omega}'\, d\mathbf{r}'\, dt', \tag{10}$$

from which it is seen that I^* is a linear functional of the radiation intensity, considered as a function of the coordinates of a point source.

This circumstance means that the adjoint equation can be successfully applied in those problems where it is required to determine certain characteristics of radiation as functions of the parameters of a point source. (We note that adjoint equations of this kind are widely used in neutron physics, which deals with similar physical processes (1, 2).)

2. Suppose that σ does not depend on frequency and that the scattering probability satisfies the symmetry condition $p(\nu,\nu';\cos\theta) = p(\nu',\nu;\cos\theta)$. Then, if in the adjoint equation for G

$$\begin{aligned} &\left\{-\frac{1}{c}\frac{\partial}{\partial t} - (\vec{\omega}\vec{\nabla}) + \alpha(\nu,\mathbf{r}) + \sigma(\nu,\mathbf{r})\right\} G(\nu_0,\vec{\omega}_0,\mathbf{r}_0,t_0;\nu,\vec{\omega},\mathbf{r},t) \\ &- \int G(\nu_0,\vec{\omega}_0,\mathbf{r}_0,t_0;\nu',\vec{\omega}',\mathbf{r},t)\, p(\nu',\nu;\cos\theta)\, \sigma(\nu,\mathbf{r})\, d\nu'\, d\vec{\omega}' \\ &= \delta(\nu-\nu_0)\delta(\vec{\omega}-\vec{\omega}_0)\delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0) \end{aligned} \tag{11}$$

we change the signs of $\vec{\omega}$ and $\vec{\omega}_0$ and interchange t and t_0, then it coincides with the direct equation. In this case the adjoint boundary conditions pass into (4), (5). In other words, in this case optical reversibility holds:

$$G(\nu,\vec{\omega},\mathbf{r},t;\nu_0,\vec{\omega}_0,\mathbf{r}_0,t_0) = G(\nu_0,-\vec{\omega}_0,\mathbf{r}_0,t;\nu,-\vec{\omega},\mathbf{r},t_0). \tag{12}$$

This assertion is also valid for polarized light, since the angular scattering matrix is symmetric [(3)]. Thus, optical reversibility is a special case of the general reciprocity relation.

3. There are problems in which it is required to determine the radiation intensity only in some part of the medium, bounded, say, by a surface S_1.

For such problems it is desirable to find boundary conditions on S_1. We shall assume that the source for I is different from zero only inside S_1.

Let us introduce an auxiliary influence function $\Gamma(\nu, \vec{\omega}, \mathbf{r}, t; \nu', \vec{\omega}', \mathbf{r}', t')$ for the region enclosed between the outer surface S and the inner surface S_1, under the additional condition that the medium lying inside S_1 is replaced by a black body. Γ satisfies conditions (5), (7) on S and the conditions

$$\Gamma(\nu, \vec{\omega}, \mathbf{r}, t; \nu' \vec{\omega}', \mathbf{r}', t') = 0 \quad \text{when} \quad \begin{cases} \mathbf{r} \in S_1, \ \mathbf{n}_1 \vec{\omega} > 0; \\ \mathbf{r}' \in S, \ \mathbf{n}_1' \vec{\omega}' < 0, \end{cases}$$

where $\mathbf{n}_1$ is the outward normal to S_1 at the point $\mathbf{r}$, and $\mathbf{n}_1'$ at the point $\mathbf{r}'$.

Using relation (8) for the region between S_1 and S, and substituting $I^*(\nu, \vec{\omega}, \mathbf{r}, t) = \Gamma(\nu, \vec{\omega}, \mathbf{r}, t; \nu', \vec{\omega}', \mathbf{r}', t')$, we obtain

$$I(\nu, \vec{\omega}, \mathbf{r}, t)\big|_{r \in S_1, \ \vec{\omega}\mathbf{n}_1 < 0} =$$

$$= \oint\limits_{\substack{S_1 \\ \vec{\omega}'\mathbf{n}_1' > 0}} \int (\mathbf{n}_1' \vec{\omega}') \, \Gamma(\nu, \vec{\omega}, \mathbf{r}, t; \nu', \vec{\omega}', \mathbf{r}', t') \, I(\nu', \vec{\omega}', \mathbf{r}', t') \, dS_1 \, d\nu' \, d\vec{\omega}' \, dt', \quad (13)$$

where $\mathbf{n}_1'$ is the outward normal at the point $\mathbf{r}'$, over which the integration is carried out. This is the required boundary condition. It permits the incoming radiation to be determined from the outgoing radiation and is therefore quite sufficient for the complete solution of the transport equation inside S_1.

In the same way we obtain the adjoint boundary condition on S_1:

$$I^*(\nu, \vec{\omega}, \mathbf{r}, t)\big|_{r \in S_1, \ \vec{\omega}\mathbf{n}_1 > 0} =$$

$$= -\oint\limits_{\substack{S_1 \\ \vec{\omega}'\mathbf{n}_1' < 0}} \int (\mathbf{n}_1' \vec{\omega}') \, \Gamma(\nu', \vec{\omega}', \mathbf{r}', t'; \nu, \vec{\omega}, \mathbf{r}, t) \, I^*(\nu', \vec{\omega}', \mathbf{r}', t') \, dS_1 \, d\nu' \, d\vec{\omega}' \, dt'. \quad (14)$$

From (13) and (14) it follows that relation (8) is also valid for a region bounded by the surface S_1.

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Note added in proof. After the present article had been sent to press, work [4] appeared, in which the author obtained the reciprocity relation (9), proceeding from the theory of Markov chains.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.