ON THE CONVERGENCE OF SEQUENCES OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS OF TWO VARIABLES

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Abstract

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ON THE CONVERGENCE OF SEQUENCES OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS OF TWO VARIABLES

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P. P. Korovkin (1) proved the theorem that from the uniform convergence of a sequence of linear positive operators $L_n(f_k; x)$ to $f_k(x)$, $k = 1, 2, 3$, on the interval $[a, b]$, where $\{f_k(x)\}_1^3$ is a Chebyshev system on this interval, it follows that $L_n(f; x)$ converges uniformly to $f(x)$, where $f(x)$ is any continuous function on the interval.

In the present note we investigate conditions for the convergence of a sequence of linear positive operators in the space of continuous functions of two variables.

Let $L_n(f(\xi, \eta); x, y)$ be a sequence of linear positive operators defined on the set of functions $f(x, y)$ continuous in the closed and bounded domain $\overline{D}$.

Theorem 1. *If, for the sequence of linear positive operators $L_n(f; x, y)$, the following conditions are satisfied:*

$$\begin{aligned} &1) \quad L_n(1; x, y) = 1 + \alpha_n(x, y);\\ &2) \quad L_n(\xi; x, y) = x + \beta_n(x, y);\\ &3) \quad L_n(\eta; x, y) = y + \gamma_n(x, y);\\ &4) \quad L_n(\xi^2 + \eta^2; x, y) = x^2 + y^2 + \delta_n(x, y), \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n$ tend uniformly to zero in the domain $\overline{D}$, then the sequence $L_n(f; x, y)$ converges uniformly to $f(x, y)$, provided $f(x, y)$ is continuous in this domain.

Proof. Put $\psi(\xi, \eta) = (\xi - x)^2 + (\eta - y)^2$. From conditions 1)–4) and the linearity of the operators it follows that

$$\begin{aligned} L_n(\psi; x, y) = L_n(\xi^2 + \eta^2; x, y) - 2x L_n(\xi; x, y) \\ - 2y L_n(\eta; x, y) + (x^2 + y^2) L_n(1; x, y) \to 0. \end{aligned} \tag{1}$$

Since the function $f(x,y)$, continuous in a closed domain, is bounded in it, $|f(x,y)| < M$, we have

$$-2M < f(\xi,\eta) - f(x,y) < 2M. \tag{2}$$

Further, for every $\varepsilon > 0$ one can indicate a $\rho > 0$ such that the inequalities

$$-\varepsilon < f(\xi,\eta) - f(x,y) < \varepsilon, \tag{3}$$

hold whenever $(\xi - x)^2 + (\eta - y)^2 < \rho^2$.

Using inequalities (2) and (3), we easily verify the validity of the inequalities

$$-\varepsilon - \frac{2M}{\rho^2}\psi(\xi,\eta) < f(\xi,\eta) - f(x,y) < \varepsilon + \frac{2M}{\rho^2}\psi(\xi,\eta) \tag{4}$$

for any two points of the region $\overline{D}$.

From relations (4) and the monotonicity of linear positive operators it follows that

$$-\varepsilon L_n(1) - \frac{2M}{\rho^2}L_n(\psi) < L_n(f) - f(x,y)L_n(1) < \varepsilon L_n(1) + \frac{2M}{\rho^2}L_n(\psi).$$

Consequently, by virtue of the first condition of the theorem and relation (1), for sufficiently large n the inequality

$$|L_n(f;x,y) - f(x,y)| < 2\varepsilon$$

will hold.

The theorem is proved.

We shall show that, under the conditions of Theorem 1, the four functions $1, x, y, x^2 + y^2$ cannot be replaced by any three functions.

Lemma. If any three continuous functions $f_k(P)$, $k = 1, 2, 3$, are given on a set from which one can select a closed line l and a point C outside it, then there exists a polynomial

$$F(P) = a_1 f_1(P) + a_2 f_2(P) + a_3 f_3(P)$$

and at least three points P_i, $i = 1, 2, 3$, such that $F(P_i) = C$. For this polynomial not all coefficients are equal to zero.

Proof. Let A and B be any distinct points on l. If the determinant of the third order, formed from the values of $f(P)$ at the points A, B, and C, is equal to zero, then the lemma is true. If, however, this determinant is different from zero, then, interchanging A and B by a continuous motion, we shall change the sign of the determinant and again obtain three points at which the mentioned determinant will be equal to zero.

Without changing the method proposed by P. P. Korovkin for the proof of Theorem 3 in [1], the following theorem can be proved.

Theorem 2. Let $f_k(P)$, $k = 1, 2, \dots, m$, be continuous functions on some connected set E. If from the relations

$$L_n(f_k; P) \rightrightarrows f_k(P), \qquad k = 1, 2, \dots, m,$$

there follows the uniform convergence of the sequence $L_n(f; P)$ to $f(P)$, where $L_n(f, P)$ is any sequence of linear positive operators and $f(P)$ is any continuous function on E, then for any triple of points $P_i \in E$, $i = 1, 2, 3$, the rank of the matrix

$$\begin{pmatrix} f_1(P_1) & f_2(P_1) & \dots & f_n(P_1) \\ f_1(P_2) & f_2(P_2) & \dots & f_n(P_2) \\ f_1(P_3) & f_2(P_3) & \dots & f_n(P_3) \end{pmatrix}$$

is equal to three.

From the lemma and Theorem 2, Theorem 3 follows immediately.

Theorem 3. It is impossible to choose three functions $f_k(x, y)$, $k = 1, 2, 3$, continuous in the region $\overline{D}$, such that from the relations

$$L_n(f_k; x, y) = f_k(x, y) + \alpha_{n,k}(x, y), \qquad \alpha_{n,k}(x, y) \rightrightarrows 0,$$

there would follow the equality

$$L_n(f; x, y) = f(x, y) + \beta_n(x, y), \qquad \beta_n(x, y) \rightrightarrows 0,$$

where $f(x, y)$ is any continuous function in $\bar{D}$, and $L_n(f; x, y)$ is any sequence of linear positive operators.

Remark. If $\bar{D}$ is a closed and bounded domain of an m-dimensional space and $L_n(f; x_1, x_2, \dots, x_m)$ is a sequence of linear positive operators defined for all functions $f(x_1, x_2, \dots, x_m)$ continuous in the domain, then Theorem 1 will be valid if, in its condition, the functions $1, x, y, x^2 + y^2$ are replaced by the functions

$$1,\ x_1,\ x_2, \dots, x_m,\ \sum_{k=1}^{m} x_k^2,$$

and Theorem 3 remains in force when, in its condition, three functions are replaced by $m + 1$ functions.

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CITED LITERATURE

1. P. P. Korovkin, DAN, 90, No. 6 (1953).

Note: Figure translations are in progress. See original paper for figures.