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Abstract

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MATHEMATICS

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APPLICATION OF THE METHOD OF GRIDS TO THE SOLUTION OF SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

(Presented by Academician N. N. Bogolyubov, 21 IX 1956)

This paper investigates conditions for the applicability of the method of finite differences to the solution of the first boundary-value problem for a system of linear partial differential equations of second order with constant coefficients, of elliptic type in n -dimensional space. With the aid of the constructed fundamental matrix, under certain additional assumptions, the existence of a solution of the finite-difference approximation to such a system is proved. Without using the “maximum principle” (in the case under consideration it is not applicable), an estimate is given for the error of the finite-difference solution of the boundary-value problem for a second-order system with constant coefficients in spaces of two and three dimensions.

Let an infinite rectangular grid be given in n -dimensional space, with equal step h in the directions of all coordinate axes. The points of the grid have coordinates that are multiples of h ; we shall denote them by $(j_1 h, \dots, j_n h) = jh$. The value of a function u at such a point will be denoted by u_j (j_1, \dots, j_n are integers). We shall also use the usual vector notation. Denote by δ the translation operator and agree to write:

$$\begin{aligned} \delta_1^{k_1} \dots \delta_n^{k_n} u(x_1, \dots, x_n) &= u(x_1 + k_1 h, \dots, x_n + k_n h) = \\ &= u[(j_1 + k_1)h, \dots, (j_n + k_n)h] = u_{j_1+k_1, \dots, j_n+k_n} = u_{j+k}^* \end{aligned}$$

The derivatives occurring in the differential equations will be replaced according to the scheme

$$\left(\frac{\partial^{l_1 + \dots + l_n} u}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} \right)_j \sim \frac{1}{h^{l_1 + \dots + l_n}} \sum_{k \in M} C_k^{(l_1, \dots, l_n)} u_{j+k},$$

where C_k are constant coefficients independent of h ; M is some finite set of integer points, for example of the form $(0, 0, \dots, 0)$, $(\pm 1, 0, \dots, 0)$, \dots , $(0, 0, \dots, 0, \pm 1)$, etc.

Consider the finite-difference system of equations with constant coefficients

$$\sum_{k \in M} A_k u_{j+k} = f_j, \quad (1)$$

which is a difference approximation to some system of linear partial differential equations of order s with constant coefficients, of elliptic type. Here A_k are given constant square matrices; u and f are column matrices.

We define the fundamental matrix for the difference system (1) by analogy with its definition for a system of differential equations. Analogously to the method by which the fundamental matrix for a system of differential equations is constructed, we construct it by Fourier expansions,

* Here $x_1 = j_1 h, \dots, x_n = j_n h$.

first, formally, the fundamental matrix for our system (1) in the form

$$g_j = \frac{1}{(2\pi h)^n} \int_{-\pi}^{+\pi} \dots (n) \dots \int_{-\pi}^{+\pi} e^{-i(j,x)} A^{-1}(x) dx \quad (2)$$

and a particular solution of this system in the form

$$u_j = h^n \sum_{l=-\infty}^{+\infty} g_{j-l} f_l, \quad \text{where } A(x) = \sum_{k \in M} e^{-i(k,x)} A_k. \quad (3)$$

It is assumed that the matrix $A(x)$ is invertible for $x \neq 0$. Depending on the order of the singularity of the matrix $A^{-1}(x)$ at zero, following Bochner's idea, we find a correction to the integral (2) so that it is always convergent. Namely, if the expansion of $A(x)$ in powers of x begins with x^m and $m > n$, then it is assumed that the matrix of the group of lower-order terms is invertible for $x \neq 0$; in this case the fundamental matrix (2) can be written as follows:

$$g_j = \frac{1}{(2\pi h)^n} \int_{-\pi}^{+\pi} \dots (n) \dots \int_{-\pi}^{+\pi} \left\{ e^{-i(j,x)} - \sum_{l=0}^{\nu-1} \frac{(-i)^l (j,x)^l}{l!} \right\} A^{-1}(x) dx; \quad (4)$$

where ν is any of the numbers $0, 1, 2, \dots, m$ such that $m - \nu < n$.

For the three-dimensional case the following two theorems are proved, generalizing the corresponding theorem of Duffin ⁽¹⁾, proved by him for the difference Laplace operator. The first of these theorems makes it possible to estimate the behavior of the fundamental matrix of a difference system of equations that is an approximation to an elliptic system of linear differential equations of second

order with constant coefficients, homogeneous with respect to the order of differentiation. The second theorem gives, for this case, an estimate of the deviation of the difference fundamental matrix from the exact fundamental matrix.

Theorem 1.

$$g_k = \begin{cases} O\left(\frac{1}{h}\right), & (k = 0), \\ O\left(\frac{1}{h|k|}\right), & (k \neq 0); \end{cases} \quad \frac{\partial^{l_1+l_2+l_3} g_k}{\partial k_1^{l_1} \partial k_2^{l_2} \partial k_3^{l_3}} = \begin{cases} O\left(\frac{1}{h}\right), & (k = 0), \\ O\left(\frac{1}{h|k|^{1+l_1+l_2+l_3}}\right), & (k \neq 0); \end{cases}$$

$$\frac{(\delta_1 - 1)^{l_1} (\delta_2 - 1)^{l_2} (\delta_3 - 1)^{l_3} g_k}{h^{l_1+l_2+l_3}} = \begin{cases} O\left(\frac{1}{h}\right), & (k = 0), \\ O\left(\frac{1}{(h|k|)^{1+l_1+l_2+l_3}}\right), & (k \neq 0). \end{cases}$$

Theorem 2. $g_k = \varphi(y) + O(h/|y|^2)$, where $\varphi(y)$ is the fundamental matrix of the differential operator of the system at the point $y = hk$, with a singularity at zero, and

$$|y| = h|k| = [(hk_1)^2 + (hk_2)^2 + (hk_3)^2]^{1/2}.$$

Next, the existence and uniqueness of a finite-difference solution of the first boundary value problem are proved. Let, in n -dimensional space with rectangular coordinate system (x_1, \dots, x_n) , some finite domain D be given, bounded by a piecewise-smooth surface Γ . Let D_h be the lattice domain corresponding to the domain D ; Γ_h the set of boundary points of the domain D_h ; D_h^* the lattice domain corresponding to any subdomain D^* lying entirely inside D .

Let an elliptic system of differential equations be given

$$\mathfrak{B}(\partial/\partial x)u(x) = f \tag{5}$$

with boundary conditions

$$u(x)|_{\Gamma} = \varphi, \tag{6}$$

where $\mathfrak{B}(\partial/\partial x)$ is a linear matrix operator of the second order, of variational type, of size $p \times p$, with constant coefficients; u, f, φ are column matrices of p functions. It is assumed that f and φ are continuously differentiable, respectively, in D and on Γ . Let the finite-difference approximation of system (5) at the point j be the system

$$\{\mathfrak{A}(\delta)u\}_j = \sum_{k \in M} A_k u_{j+k} = f_j \tag{7}$$

with boundary conditions

$$u|_{\Gamma} = \varphi|_{\Gamma_h} \quad (8)$$

(for the values of the functions on the boundary Γ_h one may take the prescribed values at the nearest points of Γ).

It is assumed that the left-hand sides of equations (7) are obtained as the “variational” equations in the investigation of the minimum of the sum $h^n \sum_{D_h} B(u, u)$, where $B(u, u)$ is a nonnegative quadratic form depending on the first difference quotients (taken forward or backward by only one step) and on the functions themselves.

It is proved that, for every vector-function w defined on the mesh, the inequality

$$h^n \sum_{D_h^*} B(w, w) \leq c_1 h^n \sum_{D_h} \sum_{k=1}^p (w^{(k)})^2 + c_2 h^n \sum_{D_h} \sum_{k=1}^p \{\mathfrak{A}^{(k)}(\delta)w\}^{2*}. \quad (9)$$

holds.

Here the constants c_1 and c_2 depend only on the domain D_h and on the shortest distance between the boundaries Γ_h^* and Γ_h , but not on h ; $\mathfrak{A}^{(k)}(\delta)w$ is the left-hand side of the k -th equation of system (7). With the aid of this inequality, the Courant-Friedrichs-Lewy theory ⁽²⁾ is carried over to prove the existence of a solution of problem (7), (8) and the convergence of this solution to the solution of problem (5), (6) (the boundary conditions being satisfied in the mean). In particular, these arguments are applicable to the equations of the theory of elasticity.

Finally, making essential use of the notion of a fundamental matrix, an error estimate for the grid method is given for problem (5), (6) in the cases of two and three dimensions. It is additionally assumed that the equations of system (5) are homogeneous in order of differentiation, and that the functions φ are three times continuously differentiable on Γ ; in this case the solution of problem (5), (6) is also three times continuously differentiable in $D \cup \Gamma$, and therefore

$$\mathfrak{A}(\delta)u = \mathfrak{B}(\partial/\partial x)u + O(h); \quad (10)$$

it is further assumed that

$$A(x) = \sum_{k \in M} A_k e^{-i(k,x)}$$

and $\mathfrak{B}(-ix)$ are invertible. Denote by ε the deviation of the solution u_h of the finite-difference problem (7), (8) from the solution $u(x)$ of the corresponding

problem (5), (6), i.e. $\varepsilon = u_h - u(x)$. We have $\mathfrak{A}(\delta)u(x) = \mathfrak{B}(\partial/\partial x)u(x) + O(h) = f - \xi h$, where $\xi = O(1)$.

Then, for each mesh point, we obtain:

$$\mathfrak{A}(\delta)\varepsilon = \xi h, \quad \varepsilon|_{\Gamma_h} = O(h) = \eta h, \quad \text{where } \eta = O(1). \quad (11)$$

Consider the auxiliary system, not connected with the boundary conditions:

$$\mathfrak{A}(\delta)w = \xi h. \quad (12)$$

* This inequality generalizes, to the case of systems of difference equations with variable coefficients, the inequality established for the difference Laplace operator in (2).

In the case of three dimensions, one of the particular solutions of this system is

$$w_j = h^3 \sum_{k \in D_h} g_{j-k}(h\tilde{\xi}_k),$$

where g is the fundamental matrix of system (12). (13)

Lemma. If $|\xi_k| \leq N$ for $k \in D_h$, then

$$\left| h^3 \sum_{k \in D_h} g_{j-k} \xi_k \right| \leq CN,$$

where the constant C does not depend on h and ξ_k , but depends only on the domain D_h . Then

$$w = O(h) = \psi h, \quad \text{where } \psi = O(1). \quad (14)$$

Consider the auxiliary function $v = \varepsilon - w$. Applying to it the operator $\mathfrak{A}(\delta)$, on the basis of (11), (12), and (14) we obtain

$$\mathfrak{A}(\delta)v = 0, \quad v|_{\Gamma_h} = \zeta h, \quad \text{where } \zeta = O(1). \quad (15)$$

Construct a vector-function Φ at the mesh points of D_h , the individual components of which would give a minimum to the sum

$$h^3 \sum_{D_h} \sum_{j=1}^3 (\Phi_{x_j}^2 + \Phi_{x_j}^2) = h^3 \sum_{D_h} B_1(\Phi, \Phi)^*; \quad \Phi|_{\Gamma_h} = v|_{\Gamma_h}. \quad (16)$$

(The “derivative” involving the value of Φ at a point lying outside D_h shall be taken to be equal to zero.) The existence of such a function is known: it will be a “mesh-harmonic” function, since the sum (16) gives rise to the difference Laplace equation.

On the basis of (15) and the maximum principle for the harmonic function Φ , we obtain

$$h^3 \sum_{D_h} B_1(\Phi, \Phi) = O(h),$$

and consequently,

$$h^3 \sum_{D_h} B(\Phi, \Phi) = O(h), \quad h^3 \sum_{D_h} B(v, v) = O(h). \quad (17)$$

Applying now to the function v the inequality

$$h^3 \sum_{D_h} \omega^2 \leq c_1 h^2 \sum_{\Gamma_h} \omega^2 + c_2 h^3 \sum_{D_h} \sum_{j=1}^3 \omega_{x_j}^2, \quad (18)$$

proved in work ⁽²⁾, on the basis of (15) and (17) we obtain

$$h^3 \sum_{D_h} v^2 = O(h).$$

On the basis of (17), (18), and (9) it is shown that

$$h^2 \sum_{D_h^*} |v_{x_i x_j \dots}|^2 = O(h) \quad (i, j, \dots = 1, 2, 3). \quad (19)$$

From (19) and one inequality of work ⁽²⁾ it follows that the difference of the values of v at two arbitrary points D_h^* is $v_1 - v_2 = O(\sqrt{h})$. Hence it is not difficult to show that $v_1 = O(\sqrt{h})$, and consequently also $v = O(\sqrt{h})$. Then, on the basis of (14) and the fact that $\varepsilon = v + w$, we obtain $\varepsilon = O(\sqrt{h})$ for all interior points. The constant contained in $O(\sqrt{h})$ depends on the dimensions of the domain D^* .

The estimate of the error of the solution at the boundary points is carried out on the average. It is shown that at these points the error will be of order $O\{(r + h)^2\}$, where r is the thickness of the boundary layer over which the averaging is carried out.

Similar error estimates are also valid for the two-dimensional case.

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2. R. Courant, K. Friedrichs, H. Lewy, Uspekhi Mat. Nauk, vol. 8, 125 (1941).

* Here Φ is understood to mean one of the components of the vector Φ .

Note: Figure translations are in progress. See original paper for figures.

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