# ON AN INTEGRAL MANIFOLD OF A SYSTEM OF DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER

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**Abstract**

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**MATHEMATICS**

**K. V. ZADIRAKA**

# ON AN INTEGRAL MANIFOLD OF A SYSTEM OF DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER

*(Presented by Academician N. N. Bogolyubov on 11 III 1957)*

Consider the system of differential equations

$$\frac{dx}{dt} = f(t, x, z, t/\mu), \qquad \mu\frac{dz}{dt} = F(t, x, z) \tag{1}$$

with initial conditions $x^0, y^0$ for $t = t_0$, where $x$ and $f$ are $n$-dimensional, while $z$ and $F$ are $m$-dimensional vectors, and the so-called degenerate ($\mu = 0$) system and the system averaged with respect to the argument $t/\mu$,

$$\frac{d\overline{x}}{dt} = f_0(t, \overline{x}, \overline{z}), \qquad \overline{z} = \varphi(t, \overline{x}), \qquad \overline{x}(t_0) = x^0, \tag{2}$$

where $z = \varphi(t, x)$ is a root of the system $F(t, x, z) = 0$;

$$f_0(t, x, z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t, x, z, \nu)\, d\nu.$$

In the absence of the argument $t/\mu$, system (1) reduces to the system considered by A. N. Tikhonov ($^2$) and I. S. Gradshtein ($^3$) on a finite interval of $t$. We shall consider the question of the existence of an integral manifold of system (1) close to the integral manifold $z = \varphi(t, x)$.

We shall assume that in the domain

$$-\infty < t < \infty, \qquad x \in G, \qquad |z - \varphi(t, x)| \le \rho, \qquad 0 < \mu < \mu^*$$

the following conditions are satisfied:

a) the vector $f$ is continuous, bounded, and satisfies a Lipschitz condition in $x$ and $z$;

b) the vectors $F$ and $\varphi$ and their derivatives with respect to all arguments up to and including the second order are continuous and bounded;

c) the second mixed derivatives of the vector $F$ with respect to $x$ and $z$ satisfy a Lipschitz condition in $z$;

d) uniformly with respect to $x \in G$, there exists the limit

$$f_0(t,x,z) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,x,z,\nu)\,d\nu;$$

e) the roots of the characteristic equation $\det\|pE - \mathcal{U}\| = 0$, where the matrix $\mathcal{U}(t,x) = F_z\big|_{z=0}$, $E$ is the identity matrix, satisfy the condition $\operatorname{Re}\{p_i(t,x)\} \le -\alpha < 0$.

**Theorem 1.** *Under these conditions one can indicate such a positive number $\mu_0$ that, for every positive number $\mu < \mu_0$, the system*

of equations (1) we have a unique integral manifold*, representable by a relation of the form $z(t,x,\mu) = \varphi(t,x) + \psi(t,x,\mu)$, in which $\psi(t,x,\mu)$, as a function of $(t,x)$, is defined in the domain $-\infty < t < \infty$, $x \in G$, and satisfies the inequalities

$$|\psi(t,x,\mu)| \le D(\mu) < \rho, \qquad |\psi(t,x',\mu) - \psi(t,x'',\mu)| \le \Delta(\mu)|x' - x''|,$$

where $D(\mu) \to 0$, $\Delta(\mu) \to 0$ as $\mu \to 0$.

**Proof.** Passing in system (1) to the new argument by means of the substitution $t = \mu\tau$, making the change $z = \varphi(\mu\tau, x) + \xi$, and separating out the linear part in the second of them, we bring system (1) to the form

$$\frac{dx}{d\tau} = \mu\Phi(\tau,x,\xi,\mu), \qquad \frac{d\xi}{d\tau} = \mathcal{U}(\mu\tau,x)\xi + Q(\tau,x,\xi,\mu), \tag{3}$$

where

$$\mathcal{U}(\mu\tau,x) = F_\xi\big|_{\xi=0}, \qquad Q(\tau,x,\xi,\mu) = Z(\mu\tau,x,\xi) - \mu\left(\frac{\partial\varphi}{\partial x}f + \frac{\partial\varphi}{\partial t}\right),$$

with

$$|Z(\mu\tau,x,\xi)| \le M|\xi|^2, \qquad Z(\mu\tau,x,0) = 0.$$

Under the conditions imposed on the vectors $f, F$ and $\varphi$, the inequalities hold

$$\mu|\Phi(\tau,x,\xi,\mu)| \le \mu L, \qquad |Q(\tau,x,\xi,\mu)| \le M|\xi|^2 + \mu N, \tag{4}$$

$$\mu|\Phi(\tau,x',\xi',\mu) - \Phi(\tau,x'',\xi'',\mu)| \le \mu A\{|x' - x''| + |\xi' - \xi''|\},$$

$$|Q(\tau, x', \xi', \mu) - Q(\tau, x'', \xi'', \mu)| \le (B\sigma + \mu C)\{|x' - x''| + |\xi' - \xi''|\}, \quad (5)$$

where $L, M, N, A, B, C$ are constant numbers; $B\sigma = \max\{B_1|\xi|^2,\ B_2|\xi|\}$, with $\sigma \to 0$ as $|\xi| \to 0$.

Consider the matrix equation

$$\frac{dU(\tau, \tau_1, x)}{d\tau} = \mathcal{U}(\mu\tau, x)U(\tau, \tau_1, x), \qquad U\big|_{\tau=\tau_1} = E. \quad (6)$$

Since the matrix $\mathcal{U}(\mu\tau, x)$ is bounded and satisfies the Lipschitz condition

$$|\mathcal{U}(\mu\tau', x) - \mathcal{U}(\mu\tau'', x)| \le \mu H|\tau' - \tau''| \qquad (H = \text{const}), \quad (7)$$

then, according to a theorem of N. Ya. Lyashenko ([4]), for the matrix $U$ the estimate holds

$$|U(\tau, \tau_1, x)| \le Ke^{-\frac{\alpha}{4}(\tau-\tau_1)} \qquad (\tau > \tau_1). \quad (8)$$

Instead of system (3), we shall consider the system

$$\frac{dx}{d\tau} = \mu\Phi_1(\tau, x, \xi, \mu),$$

$$\frac{d\xi}{d\tau} = \mathcal{U}_1(\mu\tau, x)\xi + Q_1(\tau, x, \xi, \mu), \quad (3')$$

where $\Phi_1,\ Q_1,\ \mathcal{U}_1$ coincide with $\Phi,\ Q,\ \mathcal{U}$ in the domain $-\infty < t < \infty,\ x \in G,\ |\xi| \le \rho,\ 0 < \mu < \mu^*$, and outside this domain satisfy conditions (4), (5), (7), (8) with constants $L_1,\ M_1,\ N_1,\ A_1,\ B_1,\ C_1,\ H_1,\ K_1$.

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* In speaking of an integral manifold, we mean that from the relation $z_t = z(t, x_t, \mu)$, valid at some instant of time $t_0$, there follows the validity of this relation for some $t$, as long as $x_t$ remains in the domain $G$.

Considering the recurrence relation

$$\psi_{n+1}(\tau, x) = \int_{-\infty}^{\tau} U_1(\tau, \tau - z, x)Q_1\{\tau + z;\ x_n^\tau(x);\ \psi_n(\tau + z;\ x_n^\tau(x));\ \mu\}\,dz, \quad (9)$$

where $x_n^\tau$ and $\dot{x}_n^\tau$ are solutions of the equation

$$\frac{dx}{d\tau} = \mu\Phi_1(\tau, x, \psi_n(\tau, x), \mu), \tag{10}$$

with initial values, respectively, $x^0$ and $x^*$ at $\tau = \tau_0$, and using the method of N. N. Bogolyubov ([1]), we establish the inequalities

$$|\psi_n(\tau, x)| \leq D(\mu), \qquad |\psi_{n+1}(\tau, x) - \dot{\psi}_n(\tau, x)| \leq$$

$$\leq \Delta(\mu)|x^0 - x^*| + \frac{1}{2}|\psi_n - \dot{\psi}_{n-1}| \tag{11}$$

and, in particular,

$$|\psi_n(\tau, x) - \dot{\psi}(\tau, x)| \leq \Delta(\mu)|x^0 - x^*|, \tag{12}$$

$$\|\psi_{n+1}(\tau, x) - \psi_n(\tau, x)\| \leq \frac{1}{2}\|\psi_n - \psi_{n-1}\|, \tag{13}$$

where for $\mu < \mu_0$, $D(\mu) \to 0$ and $\Delta(\mu) \to 0$ as $\mu \to 0$.

These inequalities guarantee that the functions $\psi_n$ belong to the domain of definition $\xi$ and that there exists a unique solution $\psi(\tau, x, \mu)$ of the equation

$$\psi(\tau, x, \mu) = \int_{-\infty}^{\tau} U(\tau, \tau - z, x) Q\{\tau + z;\, x^\tau(x);\, \psi(\tau + z;\, x^\tau(x));\, \mu\}\, dz,$$

which, obviously, satisfies the requirements of Theorem 1 and, as is easy to show, determines an integral manifold for the system of differential equations (3) under consideration.

**Theorem 2.** *If, in addition to the hypotheses of Theorem 1, the integral manifold of system (2) has bounded and uniformly continuous derivatives with respect to $x$ up to order $(m+1)$ inclusive in the domain $-\infty < t < \infty$, $x \in G$, $|\xi| \leq \rho$, $0 < \mu < \mu^*$, the vector $f$ has bounded and uniformly continuous derivatives with respect to $x$ and $z$ up to order $m$, and the vector $F$ up to order $(m+2)$ inclusive, then the integral manifold $z(t, x, \mu) \equiv \varphi(t, x) + \psi(t, x, \mu)$ of system (1) will have bounded and uniformly continuous derivatives with respect to $x$ up to order $m$ inclusive.*

**Proof.** Differentiating the recurrence relation (17) with respect to $x$, we have

$$\frac{\partial \psi_{n+1}(\tau, x)}{\partial x} = \int_{-\infty}^{\tau} \left[ U'(\tau, \tau - z, x) \frac{dQ}{dx_n^\tau} \frac{dx_n^\tau}{dx} + \frac{\partial U(\tau, \tau - z, x)}{\partial x} Q \right] dz. \tag{14}$$

To estimate $dx_n^\tau/dx$, consider the matrix equation

$$\frac{d}{d\tau}\frac{dx_n^\tau}{dx} = \mu\frac{d\Phi}{dx_n^\tau}\frac{dx_n^\tau}{dx}, \qquad \left.\frac{dx_n^\tau}{dx}\right|_{\tau=\tau_0} = E, \tag{15}$$

obtained by differentiating equation (10) with respect to $x$.

Since the matrix of coefficients of equation (15) satisfies the hypotheses of the theorem cited earlier ([4]), we have the estimate

$$\left|\frac{dx_n^\tau}{dx}\right| \le K_2 e^{-\frac{\alpha}{4}(\tau-\tau_0)} \qquad (\tau > \tau_0). \tag{16}$$

To estimate $U_x$, we differentiate the matrix equation (6) with respect to $x$

$$\frac{dU_x(\tau,\tau_1,x)}{d\tau} = \mathcal{U}(\mu\tau,x)U_x + \frac{\partial\mathcal{U}}{\partial x_n^\tau}\frac{dx_n^\tau}{dx}U, \qquad U_x|_{\tau=\tau_1} = 0. \tag{17}$$

Taking into account that for the matrix equation

$$\frac{dW}{d\tau} = \mathcal{U}(\mu\tau,x)W, \qquad W|_{\tau=\tau_1} = E,$$

we have the estimate

$$|W(\tau,\tau_1,x)| \le Ke^{-\frac{\alpha}{4}(\tau-\tau_1)} \qquad (W \equiv U),$$

we obtain for $U_x$ from (17)

$$U_x(\tau,\tau_1,x) = \int_0^\tau W(\tau,\theta,x)\frac{\partial\mathcal{U}}{\partial x_n^\tau}\frac{dx_n^\tau}{dx}U(\theta,\tau_1,x)\,d\theta,$$

$$|U_x(\tau,\tau_1,x)| \le \frac{8K^2K_2c_0}{\alpha}e^{-\frac{\alpha}{4}(\tau-\tau_1)}\left(\left|\frac{\partial\mathcal{U}}{\partial x_n^\tau}\right| \le c_0\right).$$

Using the conditions of Theorem 2 and the estimates (16), (18), from relation (14) we have

$$\left|\frac{\partial\psi_{n+1}(\tau,x)}{\partial x}\right| \le \int_{-\infty}^\tau \left[Ke^{-\frac{\alpha}{4}|z|}\left|\frac{dQ}{dx_n^\tau}\right|K_2e^{-\frac{\alpha}{4}|z|} + \frac{8K^2K_2c_0}{\alpha}e^{-\frac{\alpha}{4}|z|}|Q|\right]dz \le$$

$$\le \frac{4K}{\alpha}\left(a_1 + \frac{16K^2K_2c_0a_0}{\alpha}\right) = c_1 \qquad \left(|Q| \le a_0, \quad \left|\frac{dQ}{dx_n^\tau}\right| \le a_1\right).$$

Similarly we obtain the estimates

$$\left|\frac{d^l\psi_n}{dx^l}\right| \leq c_l \qquad (l = 1, 2, \dots, k).$$

Applying now to the recurrence relation (14) the method of proof of Theorem 1, we obtain for $\partial^l\psi_n/\partial x^l$ inequalities analogous to inequalities (11) and (13), ensuring uniform convergence of $\partial^l\psi_n/\partial x^l$ to $\partial^l\psi/\partial x^l$ $(l = 1, 2, \dots, k)$.

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Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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*Note: Figure translations are in progress. See original paper for figures.*