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Abstract

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EMBEDDING THEOREMS FOR ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

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Embedding theorems for real analytic functions in a real domain can be proved by using two theorems of S. M. Nikol' skii, the formulations of which are given below.

We shall say that a real function $f(x_1, \dots, x_n)$, 2π-periodic in each argument, belongs to the class $B^{(\delta_k)} H^{(r)*}_{px_k}(M)$ if $f(x_1, \dots, x_k + iy_k, \dots, x_n)$ is analytic in $x_k + iy_k$ in the strip $-\delta_k < y_k < \delta_k$, for any fixed real values of the remaining variables $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$, and if, moreover, there exists the limit

$$\lim_{y_k \to \pm\delta_k} f(x_1, \dots, x_k + iy_k, \dots, x_n) = \varphi_k(x_1, \dots, x_k, \dots, x_n)$$

such that the function $\varphi_k(x_1, \dots, x_n)$, as a function of the variable x_k, belongs to the class $H^{(r)*}_{px_k}(M)$. (For the definition of this class see ([1]) or ([2]).)

If

$$f(x_1, \dots, x_n) \in B^{(\delta_1)} H^{(r)*}_{px_1}(M), \dots, f(x_1, \dots, x_n) \in B^{(\delta_n)} H^{(r)*}_{px_n}(M),$$

then we shall say that $f(x_1, \dots, x_n) \in B^{(\delta_1, \dots, \delta_n)} H^{(r)*}_{p}(M)$. Denote by $E_{\nu_1 \dots \nu_n} f(x_1, \dots, x_n)_{L_p^*}$ the best approximation, in the metric L_p^*, of the function f by trigonometric polynomials $T_{\nu_1 \dots \nu_n}(x_1, \dots, x_n)$ of orders $\nu_1, \dots, \nu_n$, respectively, in $x_1, \dots, x_n$. By $E_{\nu_k} f_{L_p^*}$ we denote the best approximation, in the metric L_p^*, of the same function f by trigonometric polynomials $T_{\nu_k}(x_1, \dots, x_n)$ of order ν_k only in the one variable x_k.

The above-mentioned theorems of S. M. Nikol' skii, as applied to functions of several variables, can be formulated as follows:

I. If $f(x_1, \dots, x_n) \in B^{(\delta_k)} H^{(r)*}_{px_k}(M)$, then in every case, for $r < 1/p$,

$$E_{\nu_k x_k} f(x_1, \dots, x_n)_{L_p^*} \leq \frac{CM}{\nu_k^r e^{\nu_k \delta_k}}.$$

II. If $f(x_1, \dots, x_n) \in L_p^*$ and the preceding inequality is satisfied, then for $r > 1/p$ and $\delta_k > 0$

$$f(x_1, \dots, x_n) \in B^{(\delta_k)} H_{px_k}^{(r)*}(M_1).$$

The proofs of the embedding theorems rest on the following lemma:

Lemma. For any function $f(x_1, \dots, x_n) \in L_p^*$ and any p from the interval $1 < p < \infty$, the inequality

$$E_{\nu_1 \dots \nu_n} f(x_1, \dots, x_n)_{L_p^*} \leq C \sum_{k=1}^{n} E_{\nu_k} f(x_1, \dots, x_n)_{L_p^*},$$

holds, where $C = C(p, n)$ is a constant depending only on p and n.

Theorem 1. Let $1 \leq m \leq n$, $1 < p \leq p' < \infty$, $\rho = r - (n/p - m/p') > 1/p$; then, if the function $f(x_1, \dots, x_n) \in \bar{B}^{(\delta_1 \dots \delta_n)} H_{p*}^{(r)}(M)$, then in the variables $x_1, \dots, x_m$, for any fixed $x_{m+1}, \dots, x_n$, it will belong to the class $\bar{B}^{(\delta_1 \dots \delta_m)} H_{p'*}^{(\rho)}(M_1)$.

Proof. Let $2^s/\delta_k < \nu_k = [2^s/\delta_k + 1] \leq 2^s/\delta_k + 1$, and let further $T_{\nu_1 \dots \nu_n} = T_s$ be a trigonometric polynomial of order $\nu_1, \dots, \nu_n$, best for f in the metric L_p^*; then, on the basis of the lemma and of the first cited theorem of S. M. Nikol' skii, one may write:

$$\|f - T_s\|_{L_p}^* = E_{\nu_1 \dots \nu_n} f_{L_p^*} \leq C \sum_{k=1}^{n} E_{\nu_k} f_{L_p^*} \leq C_1 \sum_{k=1}^{n} \frac{M}{e^{\nu_k \delta_k} \nu_k^r} \leq C_2 2^{-rs} e^{-2^s}.$$

It follows from this that f can be represented in the form of the series converging to it in the sense of $L_{p*}^{(n)}$

$$f = T_0 + \sum_{s=1}^{\infty} (T_s - T_{s-1}) = \sum_{s=0}^{\infty} Q_s,$$

where $\|Q_s\|_{L_p^*} \leq C_3 2^{-rs} e^{-2^s}$.

Using the well-known inequality of S. M. Nikol' skii ([1])

$$\|Q_s\|_{L_p^{(m)}} \leq 2^{2n} \left(\prod_{k=1}^{m} \nu_k \right)^{1/p - 1/p'} \left(\prod_{m+1}^{n} \nu_k \right)^{1/p} \|Q_s\|_{L_p^{(n)}},$$

we obtain

$$\|Q_s\|_{L_{p'}^{(m)}} \le C_3 2^{-rs} e^{-2^s} 2^{2n} C_4 2^{s(n/p - m/p')} \le C_5 2^{-s\rho} e^{-2^s}.$$

Using the last inequality, we have

$$\left\| f - Q_0 - \sum_{s=1}^{\mu-1} Q_s \right\|_{L_{p'}^{(m)}} \le C_5 \sum_{\mu}^{\infty} 2^{-s\rho} e^{-2^s} \le C_6 2^{-\rho\mu} e^{-2^\mu}.$$

Since $\sum_{s=0}^{\mu-1} Q_s$ is a trigonometric polynomial of order

$$\nu_k = [2^{\mu-1}/\delta_k + 1]$$

in the variables $x_k,\ k = 1, \dots, m$, it follows that

$$E_{\nu_k x_k} f_{L_{p'}^{(m)}} \le E_{\nu_1 \dots \nu_m} f_{L_{p'}^{(m)}} \le \left\| f - \sum_{s=0}^{\mu-1} Q_s \right\|_{L_{p'}^{(m)}} \le C_6 e^{-2^\mu} 2^{-\rho\mu} \le C_7 e^{-\nu_k \delta_k} \nu_k^{-\rho}.$$

Thus, for any $x_k,\ k = 1, \dots, m$, we have

$$E_{\nu_k x_k} f_{L_{p'}^{(m)}} \le C_7 e^{-\nu_k \delta_k} \nu_k^{-\rho},$$

whence, on the basis of the second cited theorem of S. M. Nikol' skii, it follows that

$$f \in \bar{B}_{x_k}^{(\delta_k)} H_{p'}^{(\rho)}(M_1)$$

for any $x_k,\ k = 1, \dots, m$, i.e.,

$$f \in \bar{B}^{(\delta_1 \dots \delta_m)} H_{p'}^{(\rho)}(M_1).$$

The theorem is proved.

Theorem 2. Let some function $\psi(x_1, \dots, x_m)$ of m variables belong to $\bar{B}^{(\delta_1 \dots \delta_m)} H_{p*}^{(\rho)}(M)$, $\rho > 1/p,\ 1 < p < \infty$.

Then, whatever positive numbers $\delta_{m+1}, \dots, \delta_n$ may be, one can construct a function $f(x_1, \dots, x_n)$ of n variables possessing the properties:

1) $f \in D^{(\delta_1 \dots \delta_n)} H_{p^*}^{(r)}(M_1)$, where $r = \rho + (n-m)/p$,
2) $f(x_1, \dots, x_m, 0, \dots, 0) = \psi(x_1, \dots, x_m)$.

Proof. Let $2^s/\delta_k < \nu_k = [2^s/\delta_k + 1] \leqslant 2^s/\delta_k + 1$, $s = 1, 2, \dots$. Further, let $T_{\nu_1 \dots \nu_m} = T_s$ be the trigonometric polynomial of order $\nu_1, \dots, \nu_m$ in the variables $x_1, \dots, x_m$, best for ψ in the metric $L_{p^*}^{(m)}$. Then

$$\|\psi - T_s\|_{L_{p^*}^{(m)}} \leqslant CM \sum_{k=1}^{m} \frac{1}{\nu_k^{\rho} e^{\delta_k \nu_k}} \leqslant C_1 M 2^{-s\rho} e^{-2^s}.$$

It follows from this that ψ can be represented in the form of a series converging to it in the sense of $L_{p^*}^{(m)}$,

$$\psi = \sum_{s=0}^{\infty} Q_s,$$

where $Q_0 = T_0$, $Q_s = T_s - T_{s-1}$, $s = 1, 2, \dots$,

$$\|Q_s\|_{L_{p^*}^{(m)}} \leqslant C_2 M 2^{-s\rho} e^{-2^s}, \qquad \|Q_0\| \leqslant C_2(M + \|\psi\|).$$

Introduce for consideration trigonometric polynomials $P_\nu(x)$ of order ν with the following properties

$$P_\nu(0) = 1, \quad \|P_\nu(x)\|_{L_{p^*}} = \left(\int_{-\pi}^{\pi} |P_\nu(x)|^p dx\right)^{1/p} \leqslant \frac{A}{\nu^{1/p}},$$

where A is a constant independent of ν. As such polynomials one may take

$$P_\nu(x) = \frac{\sin(\nu + 1/2)x}{2(\nu + 1/2)\sin x/2}.$$

Define the function $f(x_1, \dots, x_n)$ by means of the series

$$f = \sum_{s=0}^{\infty} Q_s \prod_{k=m+1}^{n} P_{\nu_k}(x_k) = \sum_{s=0}^{\infty} R_s.$$

Then

$$\|R_s\|_{L_{p^*}^{(n)}} \leqslant \|Q_s\|_{L_{p^*}^{(m)}} \prod_{k=m+1}^{n} \|P_{\nu_k}\|_{L_p^*} \leqslant$$

$$\leqslant C_3 M 2^{-s\rho} e^{-2^s} A^{n-m} 2^{s(n-m)/p} \leqslant C_4 M 2^{-sr} e^{-2^s},$$

whence

$$\left\|f-\sum_{s=0}^{\mu-1} R_s\right\|_{L_{p^*}^{(n)}} \leqslant \sum_{s=\mu}^{\infty}\|R_s\|_{L_{p^*}^{(n)}} \leqslant C_4 M \sum_{s=\mu}^{\infty} 2^{-sr} e^{-2^s} \leqslant C_5 M 2^{-\mu r} e^{-2^\mu}.$$

Since $\sum_{s=0}^{\mu-1} R_s$ is a trigonometric polynomial of order $\nu_k =$

$$= [2^{\mu-1}/\delta_k + 1]$$

in $x_k,\ k = 1, \dots, n$, it follows that

$$E_{\nu_k x_k}(f)_{L_{p^*}^{(n)}} \leqslant \left\|f-\sum_{s=0}^{\mu-1} R_s\right\|_{L_p^{(n)}} \leqslant C_5 M 2^{-\mu r} e^{-2^\mu} \leqslant C_6 M/\nu_k^r e^{\nu_k \delta_k}.$$

On the basis of the theorems of S. M. Nikol' skii, hence we have $f \in B_{x_k}^{(\delta_k)} H_{p^*}^{(r)}(M_1)$ for any x_k, $k = 1, \dots, n$, and this means that

$$f(x_1, \dots, x_n) \in B^{(\delta_1 \dots \delta_n)} H_{p^*}^{(r)}(M_1).$$

As a consequence of the property of the polynomials $P_\nu(x)$,

$$f(x_1, \dots, x_m, 0, \dots, 0) = \psi(x_1, \dots, x_m),$$

and the theorem is proved.

This note arose as a result of the author' s participation in the seminar on the theory of best approximations under the direction of S. M. Nikol'skii, who at one of the seminar meetings communicated the formulation of the problem solved in the note. The author takes this opportunity to express gratitude to Prof. S. M. Nikol' skii for critical remarks and valuable advice in preparing the note.

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REFERENCES

[1] S. M. Nikol' skii, Tr. Matem. inst., 38, 279 (1951). [2] S. M. Nikol' skii, DAN, 76, No. 6 (1951).

Note: Figure translations are in progress. See original paper for figures.